Some proofs about this Discrete Fourier Transform (DFT)

↳ in contrast to impossible to actually do things like $\int_{-\infty}^{\infty} dt$ we will have finite sums of data sampled at some freq f_0

FYI: DFT means pretty much the same as FFT Fast Fourier Transform

start: grab samples (voltages) at a sequence of N equally spaced times i.e. $t = k\Delta \quad k = 0, \dots, N-1$

Remark: In common practice N is a power of 2 eg $N = 1024 = 2^{10}$ or $N = 8192 = 2^{13}$ but this is not required (although it makes it easier for the usual FFT algorithm)

The full period of data collection is $T = N\Delta$

The lowest freq we could then measure is $\frac{1}{T} = \frac{1}{N} f_0$

We will also be looking at harmonics of this fundamental $f = \frac{n}{N} f_0$. Recall that Fourier Transform had terms like

$$e^{2\pi i \overset{\frac{n}{N} f_0}{f} \overset{k\Delta}{t}} = e^{2\pi i \frac{nk}{N}}$$

Remark: we will often denote terms like this with W

Frequently (but not always) $W = e^{-2\pi i/N}$

Remark: we will try and be consistent with labels for harmonic # (n) and time # (k)

Think about the collection of sampled data $\{h_k\}$ $k = 0, N-1$ as a vector $\vec{h}$ in a N dimensional space.

Fourier Analysis is an attempt to write this vector as a linear combination of vectors that represent the various harmonics: $\vec{h} = \sum H_n \vec{e}_n$

$\vec{e}_0 = (1, 1, 1, \dots, 1)$

$\vec{e}_1 =$ 

$\vec{e}_2 =$ 

etc

To be precise the k^{TH} component of the vector $\vec{e}_n$ (denote: $\vec{e}_n(k)$) is $e^{2\pi i nk/N}$

↑ sometimes (but not always) I'll call this W^{-nk}

Important property of these $\vec{e}_n$ vectors (which are harmonics)
$\vec{e}_n$ is orthogonal (ie $\perp$) to $\vec{e}_m$ if $n \neq m$

Remark: what does orthogonal mean in a N dimensional vector space ... that the dot product is zero.
what does dot product mean with complex numbers as coordinates ... use complex conjugates.

Note: An important property of dot products was $\vec{v} \cdot \vec{v} \geq 0$ and equals zero only if $\vec{v} = 0$. If we did not use complex conjugate then $\vec{v} = (0, i, 0)$ $\vec{v} \cdot \vec{v} = i^2 = -1$
but with complex conjugate $(i) \cdot (i)^* = i(-i) = 1$
So dot product $\vec{w} \cdot \vec{v} = \sum w_k^* v_k$

Remark: The result of all of this (complex vector space with a dot product) is called by mathematicians a Hilbert Space.

Remark: Foolishly sometimes mathematicians define $\vec{w} \cdot \vec{v} = \sum w_k v_k^*$. Its no big deal but it makes mathematics books tricky to use to look up results.

Proof of orthogonality:

$$\vec{e}_n \cdot \vec{e}_m = \sum_{k=0}^{N-1} e_n(k)^* \, e_m(k) = \sum_{k=0} \left(e^{2\pi i n k/N}\right)^* \left(e^{2\pi i m k/N}\right)$$

$$= \sum_{k=0}^{N-1} \left(e^{2\pi i \frac{(m-n)}{N}}\right)^k = \sum_{k=0}^{N-1} W^k = \frac{1-W^N}{1-W}$$

I'm going to call this W (sorry)

know how to sum a geometric series?

$$\text{Now } W^N = e^{2\pi i \frac{(m-n)}{N} N} = e^{2\pi i (m-n)} = 1^{(m-n)} = 1$$

Remark: if $m = n$ then $W = 1$ and the sum formula has $\frac{0}{0}$ which is bad – but if $W = 1$

$$\sum_{k=0}^{N-1} W^k = N !!$$

Kronecker delta → by definition 0 unless $n = m$ in which case it is 1

Result: $\vec{e}_n \cdot \vec{e}_m = N \delta_{nm}$

Fouriers Trick to find H_n such that $\vec{h} = \sum H_n \vec{e}_n$

Dot both sides with $\vec{e}_m$; of all the terms in the sum only the term with $n = m$ is non-zero thus

$$\vec{e}_m \cdot \vec{h} = \sum_{n=0}^{N-1} H_n \vec{e}_m \cdot \vec{e}_n = H_m \vec{e}_m \cdot \vec{e}_m = H_m N$$

thus
$$H_m = \frac{1}{N} \vec{e}_m \cdot \vec{h} \quad (\text{eg } H_5 = \tfrac{1}{N} \vec{e}_5 \cdot \vec{h})$$
$$= \frac{1}{N} \sum_{k=0}^{N-1} e^{-2\pi i \frac{mk}{N}} h_k$$

Result:
$$h_k = \sum H_m \vec{e}_m(k) = \sum_{m=0}^{N-1} H_m e^{2\pi i \frac{mk}{N}}$$
$$H_m = \frac{1}{N} \sum_{k=0}^{N-1} e^{-2\pi i \frac{mk}{N}} h_k$$

To compare to integral form recall $\frac{m}{N} = \frac{f}{f_0} = f\Delta$

so $\frac{m}{N} k = f(k\Delta) = ft$

the difference between freq $\Delta f = \frac{f_0}{N}$ $\qquad \frac{1}{\Delta N} = \frac{1}{T}$

$$\text{so } \int H(f) e^{2\pi i f t} df \rightarrow \sum H(f) e^{2\pi i f t} \frac{f_0}{N}$$
$$\int h(t) e^{-2\pi i f t} dt \rightarrow \underbrace{\sum h(t) e^{-2\pi i f t}}_{N \cdot H_m} \Delta$$

upshot: our desired H_n is the integral $\frac{H(f)}{T}$

Remark: This is pretty clear just from units. Our DFT H has the same units as h whereas our continuous $H(f)$ has units of $[h][\text{time}]$

Remark: Just as you will find the 2πs in different places for the Fourier integral, you'll find the N in different places for the discrete Fourier sums

FYI: Mathematica likes the form with $\frac{1}{\sqrt{N}}$ in both sums

Remark: if we take our expression for h_k and evaluate it for K outside the sampling time eg $K = N + j$ the result is

$$\underbrace{2\pi m}_{e^{2\pi i m} = 1} + 2\pi m j/N$$

$$h_{N+j} = \sum_{m=0}^{N-1} H_m e^{\frac{2\pi i m (N+j)}{N}}$$

$$= \sum_{m=0}^{N-1} H_m e^{2\pi i m j/N} = h_j$$

That is by the mathematics (but perhaps not in reality) h is periodic.

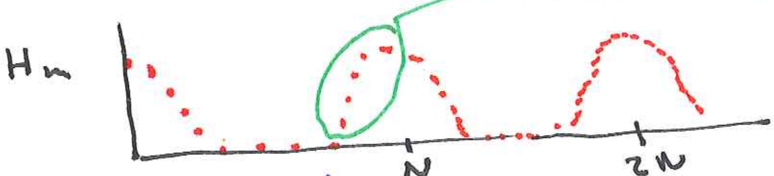
Remark: if we take our expression for H_m and evaluate it for frequencies above f_0 eg $f = \frac{(N+j)}{N} f_0$ (or $m = N+j$) the result is

$$H_{N+j} = \frac{1}{N} \sum_{K=0}^{N-1} h_K e^{-2\pi i (N+j)K/N} = \frac{1}{N} \sum h_K e^{-2\pi i j K/N}$$

$$= H_j$$

That is by the mathematics (but certainly not in reality) H_m is periodic

these are negative freq aliased up



looks good at Nyquist

Remark: The dot product of 2 vectors should be invariant to basis choice...

$$\vec{g}\cdot\vec{h} = \sum_{\ell} G_{\ell}^{*}\vec{e}_{\ell}\cdot\sum_{k} H_k\vec{e}_k = \sum_{k} G_k^{*} H_k N$$

all except one of these dot products are zero

if $\ell = k$ result is

Strange (but not uncommon) idea – what happens if you want to mix $\vec{g}$ & $\vec{h}$ together at slightly different times (this is called convolution) eg

$$\vec{g} = (a, b, c, d, e, \dots)$$
$$\vec{h} = (\alpha, \beta, \gamma, \delta, \varepsilon, \dots)$$
$$= a^{*}\cdot\gamma + b^{*}\cdot\delta + c^{*}\varepsilon + \dots$$

$\vec{g}$ is advanced by 2 compared to normal

[Eg coherence length of laser]

Notation: $\vec{g}\otimes\vec{h}(j) = \sum g_{k-j}^{*} h_k$ ← $\vec{g}$ is advanced by j

$$\vec{e}_n\otimes\vec{e}_m(j) = \sum\left(e^{2\pi i n\frac{(k-j)}{N}}\right)^{*} e^{2\pi i\frac{mk}{N}}$$

$$= e^{2\pi i\frac{nj}{N}}\sum e^{-2\pi i\frac{nk}{N}} e^{2\pi i\frac{mk}{N}} = N\delta_{nm}$$

$$\vec{g}\otimes\vec{h}(j) = \sum_{n,m} G_n^{*} H_m\, \vec{e}_n\otimes\vec{e}_m(j)$$

$$= \sum_{n} G_n^{*} H_n\, e^{2\pi i n j/N}$$

This is exactly of Fourier Transform form with amplitude $G_n^{*} H_n$

ie $\overrightarrow{\vec{g}\otimes\vec{h}} = \sum G_n^{*} H_n \vec{e}_n$