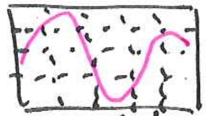
Topic: Digital Signal Processing (DSP)

subtopic: the "frequency domain"

typically an oscilloscope displays



we seek a mathematically equivalent description reporting the <u>amplitude</u> of the waves at frequency <u>f</u> that make up the signal. that is according to Fourier any signal can be thought of as the superposition of waves with various amplitudes / frequencies — we seek amplitude as a function of frequency.

It should be noted the Fourier promises an exact equivalence between V(t) & A(f) – they carry the same information and given one you can calculate the other. That is if we have a record of the microphone voltage during an hour long concert and calculate the corresponding amplitude vs frequency A(f) [more technically the Fourier Transform of the Voltage vs time] it somehow incorporates not only the notes played by the flute but <u>when</u> those notes were played.

Fourier Transform pair

"the Fourier Transform of h(t)"

$$H(f) = \int_{-\infty}^{\infty} h(t)\, e^{-2\pi i f t}\, dt$$

$$h(t) = \int_{-\infty}^{\infty} H(f)\, e^{2\pi i f t}\, df$$

Remark 1: you should be used to combining $2\pi f = \omega$
$\sin(2\pi f t) \rightarrow \sin(\omega t)$
but I'm not actually going to do that.

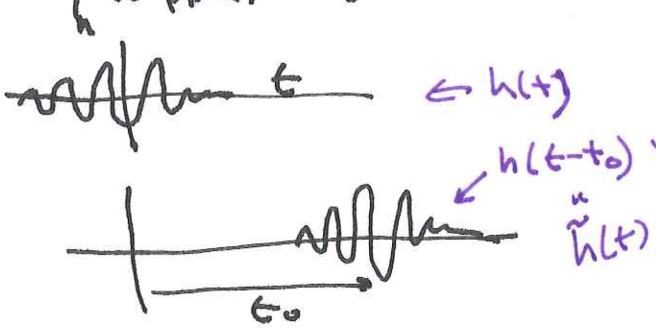
Remark 2: If you look up "Fourier Transform" in books it will show 2πs not in above — those 2πs might be $\frac{1}{\sqrt{2\pi}}$ in both integrals or $\frac{1}{2\pi}$ in one integral. All of these are saying the same thing but using different normalizations for H(f).

Remark 3: In the above we're considering variation wrt time as in $\sin(\omega t)$, but it all works just as well with variation wrt space – $\sin\left(2\pi \frac{x}{\lambda}\right)$

"wave number k"

So one can make x-k Fourier Transform pairs also

Q: How does the Fourier Transform change if the sound happens at t_0 rather than 0?



$$\tilde{H}(f) = \int_{-\infty}^{\infty} h(t-t_0)\, e^{-2\pi i f t}\, dt$$

$$= \int_{-\infty}^{\infty} h(t-t_0)\, e^{-2\pi i f (t-t_0)}\, dt\; e^{2\pi i f t_0}$$

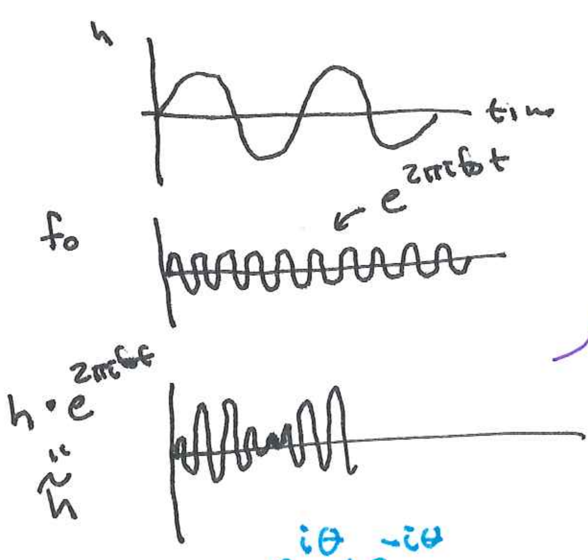
$$= H(f)\, e^{2\pi i f t_0}$$

Note: the phase of H affects time sound present

Very commonly "spectrum" [amplitude vs frequency] plots $|H(f)|^2$ [Note could not plot $H(f)$ even if desired as that's a complex number & y axis is just a real number]

$|H(f)|^2$ is usually related to something like the sound energy at f

Q: How does the Fourier Transform change if the original signal is multiplied by another (higher) frequency?



$$\tilde{H}(f) = \int_{-\infty}^{\infty} h(t)\, e^{2\pi i f_0 t}\, e^{-2\pi i f t}\, dt$$

$$= \int_{-\infty}^{\infty} h(t)\, e^{-2\pi i (f-f_0) t}\, dt$$

$$= H(f-f_0)$$

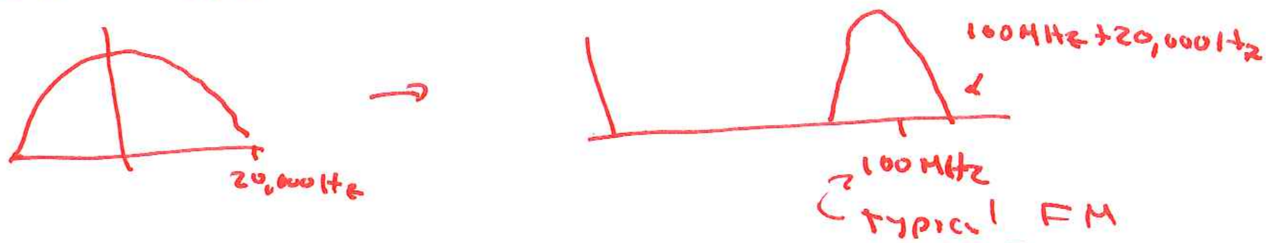
Note: multiplying shifts the Fourier transform by f_0. [the shift would be to $f+f_0$ if used $e^{-2\pi i f_0 t}$ and real sources like sin & cos involve both]

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Remark: Heterodyne - if have a signal (eg voice) with freq up to 20,000 Hz can convert it to a higher freq (with Fourier Transform simply translated by fo] by multiplying it with that higher freq "carrier freq"

Eg

spectrum of voice



Reverse also works - down convert a high freq to lower freq

Remark: I've shown above positive & negative freq. While negative freq makes no sense to physics mathematically they are part of Fourier Transform & can not be neglected. For the Fourier Transform of a real ($\mathbb{R}$) signal:

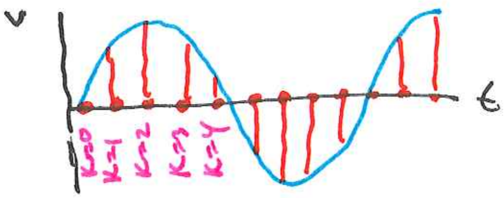
$$H(-f) = \int h(t)\, e^{+2\pi i f t}\, dt = \left[\int h(t)\, e^{-2\pi i f t}\right]^*$$

$$= H^*(f)$$

complex conjugate $(4+5i)^* = 4-5i$

So the negative freq have the same amplitude as the "normal" positive freq

We've been writing voltages as continuous functions of time but DSP will involve a sequence of snapshots of the voltage. [ie we will run the ADC at some frequency f_0] (Note: for CD sound the ADC runs at 44.1 kHz)



Voltage is recorded at a sequence of times $= k\,\Delta$

integer (k)

time between samples: $f_0 = \frac{1}{\Delta}$

The integrals of our Fourier Transforms will become sums:

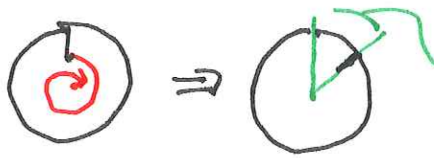
$$H(f) = \int h(t)\, e^{-2\pi i f t}\, dt \longrightarrow \sum_k h_k\, e^{-2\pi i f \Delta k}$$

h_k ie $h(k\Delta)$

Note: we will be concerned with freq below sampling freq so $f\Delta = \frac{f}{f_0} < 1$. We will find it convenient to write this fraction as something like $\frac{n}{N}$ where $n<N$. We have a lot to prove about how all of this works — wait for next time!

Important Problem with DSP: aliasing. Or why the wagon wheels may seem to go backwards in movies.

Consider a rotating object — rotating at a freq a bit above the sampling freq. During the time Δ it completes a bit more than a full rotation

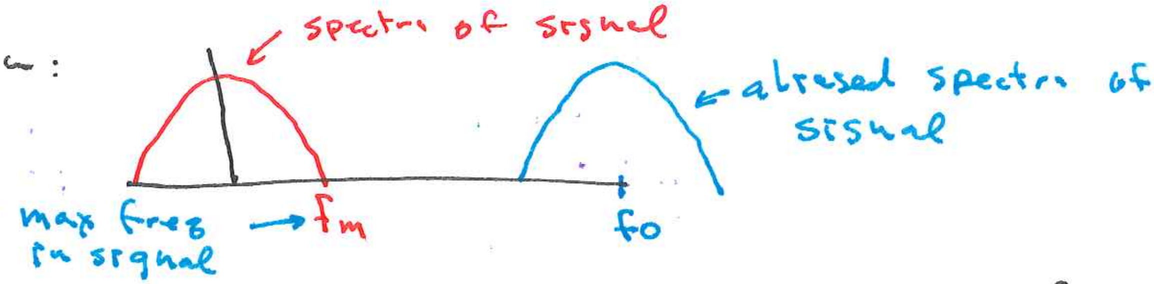


apparent rotation $= 2\pi(f\Delta - 1)$ ← $f_0\Delta = 1$

$$= 2\pi\underbrace{(f - f_0)}_{\text{apparent freq}}\Delta$$

Because of aliasing high freq motion appears as $f - f_0$. [or $f - 2f_0$ for even faster motion etc]

Problem:



negative freq can up alias & become confused with actual freq.

Require: $\underbrace{f_m < \frac{f_o}{2}}_{\text{Nyquist freq}}$ to avoid this problem

↳ Nyquist freq ← IMPORTANT!

Conclude: in order to reproduce signal that has max freq of f_m sample at $\Delta = \frac{1}{(2f_m)}$

In order to reproduce sounds heard by humans (up to 20,000 Hz) need to sample at > 40,000 Hz

(CD audio: 44.1 kHz 16 bit)