It would be a shame to leave stellar structure without taking the opportunity to solve some differential equations with Mathematica. However our textbook equations include truly complex functions for opacity and energy generation. And Swihart's practical (but in practice messy) technique of guessing a possible stellar atmosphere, solving the differential equations numerically only to not meet the boundary conditions at the origin, followed by re-guessing is not useful in real life.

So lets look at a much simpler problem, still of astrophysical interest, that presents its own not uncommon technical difficulty.

Let's consider a spherically symmetric gaseous star in thermal equilibrium, i.e., at a fixed temperature. . . no energy generation or opacity issues to make things complicated.

$$P = \frac{\rho}{m} kT \tag{1}$$

$$\frac{dP}{dr} = -\rho g = -\rho \frac{GM(r)}{r^2} \tag{2}$$

$$\frac{dM}{dr} = 4\pi r^2 \rho \tag{3}$$

Solving the second equation for $M(r)$, and plugging that into the third equation yields:

$$\frac{d}{dr}\left[-\frac{r^2}{G\rho}\frac{dP}{dr}\right] = \frac{d}{dr}\left[-r^2\frac{kT}{Gm}\frac{1}{\rho}\frac{d\rho}{dr}\right] = 4\pi r^2 \rho \tag{4}$$

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{kT}{4\pi Gm}\frac{d\ln(\rho)}{dr}\right] = -\rho \tag{5}$$

It is wise to, if possible, convert to dimensionless quantities before solving differential equations. Towards that goal, we define the dimensionless ratio $\rho/\rho_0 = \rho'$, where ρ_0 is the density at $r = 0$. The constant

$$\frac{kT}{4\pi G m \rho_0} = L^2$$

has the units of length squared, so we define unitless $r' = r/L$. (L is a multiple of the Jean's length, which measures where gravitational self-attraction can overcome gas pressure. For the center of the Sun, $L \approx 30,000$ km or 5% of the Sun's radius; for a giant molecular cloud L might be a light year.) The resulting equation is

$$\frac{1}{r'^2}\frac{d}{dr'}\left[r'^2\frac{d\ln(\rho')}{dr'}\right] = -\rho' \tag{6}$$

Since $\rho' = \rho/\rho_0$, $\rho'(0) = 1$; since g is zero at the center of the sphere Eq. 2 gives: $\frac{d}{dr}\rho'(0) = 0$. In what follows we drop the primes.

If you try to solve this straight-forwardly with Mathematica, you get an error:

```
NDSolve[{1/r^2 D[r^2 D[Log[rho[r]],r],r]==-rho[r],rho[0]==1,rho'[0]==0},rho,{r,0,10}]
```

```
                                    1
Power::infy: Infinite expression -- encountered.
                                    0.

Infinity::indet: Indeterminate expression 0. ComplexInfinity encountered.

NDSolve::ndnum: Encountered non-numerical value for a derivative at r == 0..
```

Mathematica is not happy with $r = 0$ and $1/r$, but if we can start slightly beyond $r = 0$, Mathematica will be OK. We can find the power series that solves the differential equation near $r = 0$

If we try:

$$\rho(r) = 1 + ar^2 + br^4$$

Mathematica can show us the results:

```
Series[1/r^2 D[r^2 D[Log[1+a r^2 +b r^4],r],r],{r,0,4}]

              2           2        3                4       5
6 a + (-10 a   + 20 b) r   + (14 a   - 42 a b) r   + O[r]
```

Since this must equal $-\rho$, we have $a = -1/6$, $b = 1/45$. So the function

```
f[r_]=1-r^2/6+r^4/45
```

should be a good approximation of ρ near $r = 0$.

```
solution=NDSolve[{1/r^2 D[r^2 D[Log[rho[r]],r],r]==-rho[r],
rho[0.01]==f[.01],rho'[0.01]==f'[.01]},rho,{r,0.01,10}]

Plot[{f[r],Evaluate[rho[r]/.First[solution]]},{r,0.01,2}]
```

This plot should show that $f(r)$ looks identical to ρ for small r.

Plot ρ from .01 to 10; the result should look fine. However there is a significant problem. Add to your `NDSolve` the differential equation for $M(r)$ (Eq. 3) and expand the range to $r = 100$ Note that to find the initial value $M(.01)$ you will need to integrate this differential equation using $\rho = f(r)$. Plot $M(r)$ from .01 to 100. What is the significant problem?

It turns out there is a nice (but non-physical) exact solution to our ρ differential equation, of the form:

$$\rho(r) = \frac{C}{r^2}$$

By plugging this option into the differential equation Eq. 6, find the value of the constant C that works. What exactly is wrong with this solution?

Again expand the range of solution to $r = 1000$. Plot together $M(r)$ from the differential equation solution and the $M(r)$ you would get by integrating the exact solution you found above; this should suggest the isothermal sphere has no edge.

Homework

Provide a concise write up that includes the plots and whats underlined in the above text. If you are working together, feel free to submit one answer with several names attached. (Note: the definition of 'working together' is that any group member can describe what's going on in the solution.)

Comment

It is fun to see that assumptions seemingly unrelated to those used above quickly produce exactly the same differential equation. However, the starting point is Poisson's equation which first occurs in PHYS 341. The idea is to use the exact analogy between electric fields and gravitational fields, and find Gauss' law for gravity. (Coulomb's and Newton's laws are both inverse square, but with constants named differently: $\epsilon_0 \to -1/(4\pi G)$, the minus sign turning the repulsive electric force into an attractive gravitational force.) In the below φ is the electric potential (a.k.a. voltage) and ϕ is the gravitational potential.

$$\begin{aligned} F &= \frac{q_1 q_2}{4\pi\epsilon_0 r^2} & \qquad F &= \frac{-G m_1 m_2}{r^2} \\ \mathbf{E} &= \mathbf{F}/q & \mathbf{g} &= \mathbf{F}/m \\ \mathbf{E} &= -\boldsymbol{\nabla}\varphi & \mathbf{g} &= -\boldsymbol{\nabla}\phi \\ \boldsymbol{\nabla}\cdot\mathbf{E} &= \rho/\epsilon_0 & \boldsymbol{\nabla}\cdot\mathbf{g} &= -4\pi G\rho \\ \nabla^2\varphi &= -\rho/\epsilon_0 & \nabla^2\phi &= +4\pi G\rho \end{aligned}$$

Using spherical coordinates, the Laplacian (∇^2) in the gravitational Poisson's equation yields:

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\phi}{dr}\right] = 4\pi G\rho \tag{7}$$

Now invoking Boltzmann's distribution

$$\rho = \rho_0 \exp(-m\phi/kT) \qquad \text{or} \qquad -\frac{kT}{m}\ln(\rho/\rho_0) = \phi \tag{8}$$

Plugging ϕ into Eq. 7 yields:

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\,\frac{kT}{4\pi Gm}\,\frac{d}{dr}\ln(\rho/\rho_0)\right] = -\rho \tag{9}$$

Using the dimensionless version of density: $\rho' = \rho/\rho_0$:

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{kT}{4\pi G m \rho_0}\frac{d}{dr}\ln(\rho')\right] = -\rho' \tag{10}$$

Using the dimensionless version of r brings us back to Eq. 6