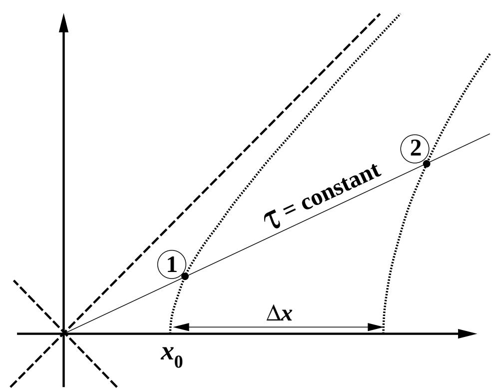
Uniform Acceleration II

Recall: The expressions for constant proper accelerated motion ($\alpha = g/c = c/x_0$) in terms of proper time τ (i.e., time measured with a clock that moves with the object, and hence subject to time dilation).

$$\begin{aligned} x &= x_0\sqrt{1+(\alpha t)^2} = x_0 \cosh(\alpha\tau) \\ ct &= x_0 \sinh(\alpha\tau) \\ \beta &= \frac{\alpha t}{\sqrt{1+(\alpha t)^2}} = \tanh(\alpha\tau) \\ \gamma &= \sqrt{1+(\alpha t)^2} = \cosh(\alpha\tau) \end{aligned}$$

The aim today is to produce a new set of coordinates $(\Delta x, c\tau)$ for space based on a frame attached to a "rigidly" accelerating object. We begin by picking an origin for our new coordinates at x_0. Our time coordinate τ will be the proper time for this origin. We apply this time to every instant in the various instantaneous rest frames. Our spatial coordinate Δx is measured from our origin.



We can find the inertial coordinate frame location (x, ct) of the moving origin in terms of its proper time:

$$\begin{aligned} x_1 &= x_0 \cosh(\alpha\tau) = x_0 \cosh(c\tau/x_0) \\ ct_1 &= x_0 \sinh(\alpha\tau) = x_0 \sinh(c\tau/x_0) \end{aligned}$$

A general point is reached by "scaling out" along a $t' = 0$ line:

$$\begin{aligned} x_2 &= \frac{x_0+\Delta x}{x_0}\, x_1 = (x_0+\Delta x)\cosh(c\tau/x_0) \\ ct_2 &= \frac{x_0+\Delta x}{x_0}\, ct_1 = (x_0+\Delta x)\sinh(c\tau/x_0) \end{aligned}$$

Note that the form of these equations is identical with those for constant proper acceleration, but with a different argument for the hyperbolic functions. So the world line of a particle 'at rest' in these new coordinates (i.e., Δx=constant) is itself undergoing constant proper acceleration when viewed from the inertial frame. Such a particle will have a proper acceleration, dependent on Δx, given by $\alpha_2 = c/(x_0 + \Delta x)$ and it will reach this same event in a proper time τ_2 such that $\alpha_2 \tau_2 = c\tau/x_0$. As a result clocks at rest in the accelerated coordinates (displaying proper time) will not agree with the coordinate time[1] τ:

$$\tau_2 = \frac{x_0 + \Delta x}{x_0} \tau = \eta\tau$$

So clocks 'above' the origin (i.e., $\Delta x > 0$), will run faster than coordinate time.

The inverse transformation equations are easily found:

$$\begin{aligned} \Delta x &= \sqrt{x^2 - (ct)^2} - x_0 \\ c\tau &= x_0 \tanh^{-1}(ct/x) \end{aligned}$$

(The transverse coordinates y and z are invariant, so I will often not record their simple behavior.)

To transform vectors between systems we need the matrix of partial derivatives:

$$\frac{\partial x^{\text{inert}}}{\partial x^{\text{accel}}} = \begin{bmatrix} \frac{\partial x}{\partial \Delta x} & \frac{\partial x}{\partial c\tau} \\ \frac{\partial ct}{\partial \Delta x} & \frac{\partial ct}{\partial c\tau} \end{bmatrix} = \begin{bmatrix} \cosh(c\tau/x_0) & (1 + \Delta x/x_0) \sinh(c\tau/x_0) \\ \sinh(c\tau/x_0) & (1 + \Delta x/x_0) \cosh(c\tau/x_0) \end{bmatrix} = \begin{bmatrix} \cosh & \eta \sinh \\ \sinh & \eta \cosh \end{bmatrix} = \lambda$$

where $\eta = 1 + \Delta x/x_0$.

To find the metric tensor in the accelerated system $g_{\mu'\nu'}$, we must transform the metric tensor in the inertial system: $g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} = \lambda^{\mathrm{T}} g_{\mu\nu} \lambda$$

The result is:

$$g_{\mu'\nu'} = \begin{bmatrix} 1 & 0 \\ 0 & -\eta^2 \end{bmatrix}$$

The inverse of this matrix is:

$$g^{\mu'\nu'} = \begin{bmatrix} 1 & 0 \\ 0 & -1/\eta^2 \end{bmatrix}$$

Note that in any frame light must travel on a null geodesic. Thus

$$0 = ds^2 = g_{\mu'\nu'} dx^{\mu'} dx^{\nu'} = (d\mathbf{r}^2 - \eta^2 c^2 d\tau^2)$$

We can conclude that speed of light in the accelerated coordinates is $c\eta$, which is larger than c for $\Delta x > 0$ and approaches zero for $\Delta x \to -x_0$. (There is no physical requirement for the coordinate speed of light to be c or even have the units of m/s.)

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[1] It is of course possible to design a clock to display coordinate time at a location Δx, but, for example, a light clock a distance Δx above the origin will seem to run fast because of the faster speed ot light up there.

The inverse of the λ matrix is also needed to transform contravariant tensors. It is easy to find the inverse of 2×2 matrices:

$$\lambda^{-1} = \begin{bmatrix} \cosh & -\sinh \\ (-1/\eta)\sinh & (1/\eta)\cosh \end{bmatrix} = \frac{\partial x^{\text{accel}}}{\partial x^{\text{inert}}}$$

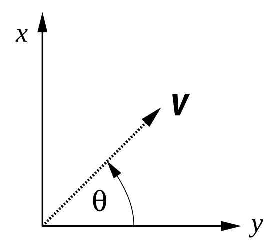
1. Directly calculate λ^{-1} by taking the derivatives of the accelerated coordinates w.r.t. the inertial coordinates.

2. Note that in the accelerated system, the metric tensor g is independent of of τ and only depends on position via η. That is only $g_{44,1} \neq 0$. Calculate the non-zero Christoffel symbols.

In the inertial frame motion is very simple: objects move at constant velocity. Thus

$$\begin{aligned} x &= x_s + v_{x0}t = x_s + v\sin\theta\ t \\ y &= v_{y0}t = v\cos\theta\ t \end{aligned}$$

(Since the system is homogeneous in the y direction, we don't bother with a "y_0" term. The starting position for x, x_s will generally be near the origin x_0.) Note that since x is the "vertical" we have an odd-looking coordinate system:



In the below *Mathematica* code, we are using the length units of light-years and time units of years. Thus, $c = 1$ and it turns out that for a proper acceleration of $g = 9.8\text{m/s}^2$, $\alpha \approx 1$. A time interval of 30 seconds would be about 10^{-6} years. Since light travels at about 1 foot per nanosecond and three nanoseconds is about 10^{-16} year, a kilometer would be about 10^{-13} light-year. The speed of sound is about $10^{-6}c$. Thus a realistic ballistic problem (a few-mach projectile with an aim-point a few kilometers away, and travel times near a minute) will produce "small" initial conditions. We should start to see relativistic effects as we move towards "unit-sized" times, velocities, and distances. The below *Mathematica* code sets up a ballistic problem.

```
x0=1

dx[x_,ct_]=Sqrt[x^2-ct^2]-x0                ...define transformation
ctau[x_,ct_]=x0 ArcTanh[ct/x]
x[dx_,ctau_]=(x0+dx) Cosh[ctau/x0]
ct[dx_,ctau_]=(x0+dx) Sinh[ctau/x0]

xs1=x0                                       ...fire cannon
```

```
v1=3 10^-6
theta1=Pi/4
ParametricPlot[{v1 Cos[theta1] t,dx[xs1+v1 Sin[theta1] t,t]},{t,0,5 10^-6},
AspectRatio->Automatic]
ParametricPlot[{ctau[xs1+v1 Sin[theta1] t,t],dx[xs1+v1 Sin[theta1] t,t]},{t,0,5 10^-6}]
```

The first `ParametricPlot` command displays the trajectory: the trace of (y, x) locations visited by the projectile. Note that the x (actually Δx) location of the projectile is calculated just by transforming a uniformly-moving inertial location. The second `ParametricPlot` command displays the $x(t)$ in the accelerated system (i.e., actually $\Delta x(c\tau)$). Again the calculation is made simply by transforming uniform inertial motion to complex τ and Δx behavior.

It is interesting to compare this transformed-motion to the usual non-relativistic results:

$$\begin{aligned} x &= x_s + v_{x0}t - \frac{1}{2}gt^2 = x_s + \beta_{x0}\, ct - \frac{1}{2}\,\frac{1}{x_0}\,(ct)^2 \\ y &= v_{y0}t = \beta_{y0}\, ct \end{aligned}$$

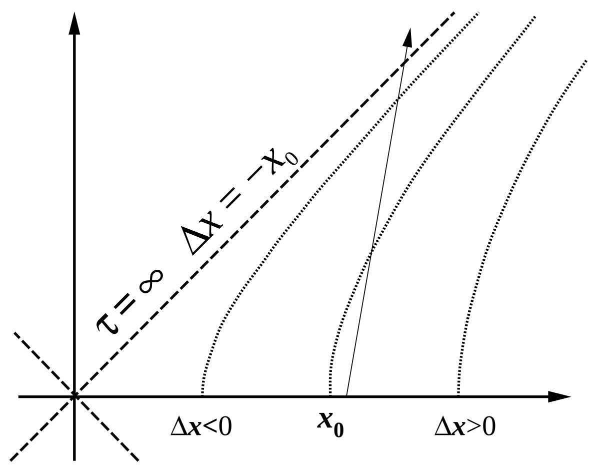
The (initial) velocity in the accelerated system can be calculated:

$$\begin{aligned} \frac{d\Delta x}{dc\tau} &= \frac{\frac{d\Delta x(x(t),t)}{dt}}{\frac{dc\tau(x(t),t)}{dt}} \\ \frac{dy}{dc\tau} &= \frac{\frac{dy}{dt}}{\frac{dc\tau(x(t),t)}{dt}} \end{aligned}$$

```
vdx[xs_,v_,theta_,t_]=D[dx[xs+v Sin[theta] t,t],t]/D[ctau[xs+v Sin[theta] t,t],t]
vdy[xs_,v_,theta_,t_]=v Cos[theta]/D[ctau[xs+v Sin[theta] t,t],t]
```

3. Establish your own initial conditions for uniform motion in the inertial frame and `ParametricPlot` the results as seen in the accelerated system. Find the equivalent initial velocity in the accelerated frame and see if the Intro Physics equations closely follow the exact result. `Show` both plots together. `PSPrint` the results. Establish relativistic initial conditions (muzzle velocities near c, ranges near a light year) and `ParametricPlot` the results.

Note that every uniformly moving object in the observable universe (i.e., $x > 0$) must eventually past through the line "cone".



The light cone is an usual place as seen in the accelerated coordinate system; I'll call it the horizon. Note that on the light cone $ct/x = 1$, and since $\tanh^{-1}(1) = \infty$, $\tau = \infty$. Thus while a clock aboard a particle crossing the light cone increments normally (in proportion to t), as seen in the accelerated frame it takes and infinite time to cross the horizon. The horizon is all at a distance $\Delta x = -x_0$. Thus a particle approaching the horizon, travels a finite distance, but takes an infinite τ to do so. Hence the coordinate velocity must approach zero. Make sure you follow your relativistic projectile to times when it approaches the horizon.

4. The path of light is bent in the accelerated system. Produce a `ParametricPlot` showing this. Note that if the x component of the velocity of light is less than c, light is also sucked into the horizon. But if light is sucked to the horizon, it must do so (like any other object) with a vanishingly small speed. As stated above the coordinate velocity of light (i.e., $d\Delta x/d\tau$) depends on position and is zero at the horizon. Since we know the speed of light measured with local clocks and meter sticks must be c, this reinforces what was stated earlier: coordinate τ is not what local clocks read as time except at $\Delta x = 0$; rather τ is "the king's" time enforced through out the realm.

We know that "$F = ma$" is not the correct equation of motion in general coordinate systems. Rather particles follow geodesics:

$$\frac{d\beta^\sigma}{ds} = -\Gamma^\sigma_{\mu\nu}\beta^\mu\beta^\nu$$

where ds is the proper time:

$$ds^2 = -g_{\mu\nu}dx^\mu dx^\nu = \eta^2 c^2 d\tau^2 - d\mathbf{x}^2 = c^2 d\tau^2 \left[\eta^2 - \left(\frac{d\mathbf{x}}{dc\tau}\right)^2\right] = c^2 d\tau^2 \left(\eta^2 - (\mathbf{u}/c)^2\right) = \left(\frac{cd\tau}{\gamma^*}\right)^2$$

and β^μ is the velocity 4-vector:

$$\beta^\mu = \frac{dx^\mu}{ds} = \gamma^* \frac{dx^\mu}{cd\tau} = \gamma^*(\mathbf{u}/c, 1)$$

Note that as usual:

$$\beta^\mu\beta_\mu = \beta^\mu\beta^\nu g_{\mu\nu} = \gamma^{*2}(\mathbf{u}^2/c^2 - \eta^2) = -1$$

We need this geodesic equation written in terms of coordinate time derivatives not proper time. (We do this in general, calling the time coordinate t, denoting ct derivatives with a dot, and dropping the $*$ on γ.) The four equations of the geodesic equation are:

$$\begin{aligned}\gamma(\dot{\gamma}u^i/c + \gamma\dot{u}^i/c) &= -\Gamma^i_{\mu\nu}\beta^\mu\beta^\nu \\ \gamma\dot{\gamma} &= -\Gamma^4_{\mu\nu}\beta^\mu\beta^\nu\end{aligned}$$

$$\begin{aligned}\gamma^2\dot{u}^i/c &= u^i/c\,\Gamma^4_{\mu\nu}\beta^\mu\beta^\nu - \Gamma^i_{\mu\nu}\beta^\mu\beta^\nu \\ a^i &= u^i/c\,\Gamma^4_{\mu\nu}u^\mu u^\nu - \Gamma^i_{\mu\nu}u^\mu u^\nu\end{aligned}$$

5. You calculated the required Christoffel symbols in **2** (also see Eqs. 8.47). Substitute your Christoffel symbols results into the geodesic equation and compare to Eqs. 8.48. Note: we found the motion in the accelerated frame by transforming coordinates for the simple inertial motion. We could have started with this geodesic equation and found the motion directly by solving the differential equation. Clearly in the usual case particle motion is not known in any frame, and solving the geodesic equation is the only available approach.