Basic Optics: Radiance

Astronomy 525

Lecture 06

Outline

- The Radiance Theorem
- Basic Radiance
- Abbe's Sine Condition
- Étendue
- Plate scales: re-imaging of pixel

- Reference: Boyd, R.W. "Radiometry and the Detection of Optical Radiation" 1983 John Wiley & Sons, Inc. Chapters 2 and 5

Radiance Theorem

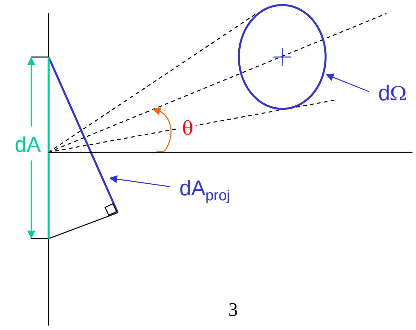
Let $d^2\Phi$ be the power (Watts) emitted into a solid angle $d\Omega$ by a source of element of projected area dA_{proj}. Then, the radiance, L, is defined by:

$$L = d^2\Phi/\{dA_{proj}\, d\Omega\} \qquad (W/m^2/sr)$$

Units of intensity

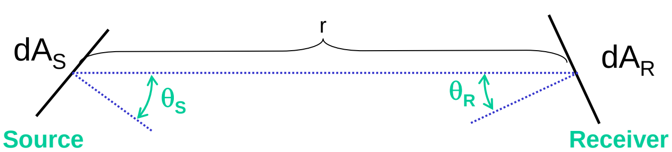
where:

$$dA_{proj} = dA \cos\theta$$



Radiance Theorem: I

The radiance is conserved through a loss-less optical system.



$d\Omega_S$ = solid angle of dA_R at dA_S
= $\{dA_R \cdot \cos\theta_R\}/r^2$ (1)

$d\Omega_R$ = solid angle of dA_S at dA_R
= $\{dA_S \cdot \cos\theta_S\}/r^2$ (2)

The power, $d^2\Phi$ transferred from dA_S to dA_R is:

$d^2\Phi = L_S \cdot (dA_S \cos\theta_S)\, d\Omega_S$ (3)

By the definition of the radiance, L_S

Radiance Theorem: II

The radiance, L_R measured at dA_R (in the same direction) is:

$$L_R = d^2\Phi/\{dA_R \cos\theta_R \, d\Omega_R\}$$

where $d\Omega_R$ is given by equation (2) above, since the flux leaves dA_R in a solid angle equal to that from which it arrived.

Using equations (1), (2), and (3) above yields: $L_R = L_S$

$$L_R = \frac{d^2\Phi}{dA_R \cos\theta_R d\Omega_R} = \frac{d^2\Phi}{dA_R \cos\theta_R \dfrac{dA_S \cos\theta_S}{r^2}}$$
$$= \frac{d^2\Phi}{d\Omega_S dA_S \cos\theta_S}$$
$$= L_S$$

Radiance Theorem: III

As a side result, we can show that it is possible to adopt the point of view of either the source or receiver when performing radiometric calculations.

Consider:

$$d^2\Phi = \mathbf{L_S\{dA_S \cos\theta_S\}\, d\Omega_S}$$
$$= L_S \cos\theta_S \, dA_S \{dA_R \cos\theta_R/r^2\}$$
$$= L_S \, dA_R \cos\theta_R\{dA_S \cos\theta_S/r^2\}$$
$$= \mathbf{L_S\{dA_R \cos\theta_R\}\, d\Omega_R}$$

That is, we can think of the power we would measure in two ways:

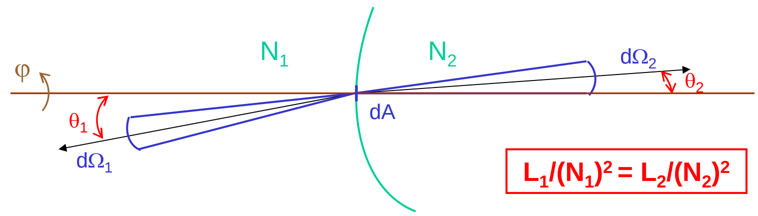
(1) From the source point of view: $d^2\Phi \propto dA_{proj}(\text{source})\, d\Omega_S$

of receiver

(2) From the receiver point of view: $d^2\Phi \propto dA_{proj}(\text{receiver})\, d\Omega_R$

of source

Basic Radiance: I



Suppose we have a beam of radiance, L, passing through a medium with refractive index N_1, falling onto dA from solid angle $d\Omega_1$ inclined at θ_1 w.r.t. dA, then the power passing through dA is given by:

$$d^2\Phi = L_1 dA \cos\theta_1\, d\Omega_1$$

We would like to find θ_2 and Ω_2 in terms of find θ_1 and Ω_1. Using polar coordinates, with the axis normal to dA, we have:

$$d\Omega_1 / d\Omega_2 = (\sin\theta_1\, d\theta_1 d\varphi_1)/(\sin\theta_2\, d\theta_2 d\varphi_2) \qquad (1)$$

Basic Radiance: II

From Snell's law we have: $d\varphi_1 = d\varphi_2$ (pick your plane of propagation)
and $N_1 \sin\theta_1 = N_2 \sin\theta_2$

so that (differentiating):

$$N_1 \cos\theta_1 d\theta_1 = N_2 \cos\theta_2 d\theta_2$$

Using equation (1) above, we then have:

$$d\Omega_1 / d\Omega_2 = (N_2/N_1)^2 (\cos\theta_2)/(\cos\theta_1)$$

The radiance of the refracted beam is then:

$$\begin{aligned} L_2 &= d^2\Phi/(dA \cos\theta_2\, d\Omega_2) \\ &= L_1\, dA \cos\theta_1\, \Omega_1/(dA \cos\theta_2\, \Omega_2) \\ &= L_1 \cdot (N_2/N_1)^2 \end{aligned}$$

$$\Rightarrow \qquad \boxed{L_1/(N_1)^2 = L_2/(N_2)^2}$$

Basic Radiance: III

Intuitive Proof of the Basic Radiance Theorem:

$$d^2\Phi = L_1\, d\Omega_1\, dA = L_2\, d\Omega_2\, dA$$

small angles on axis: $\sin\theta \approx \theta \Rightarrow d\Omega \propto \theta^2$

$$\Rightarrow L_1\, \theta_1^2\, dA \approx L_2\, \theta_2^2\, dA$$

Snell's Law: $N_1\theta_1 \approx N_2\theta_2 \Rightarrow N_1/N_2 \approx \theta_2/\theta_1$

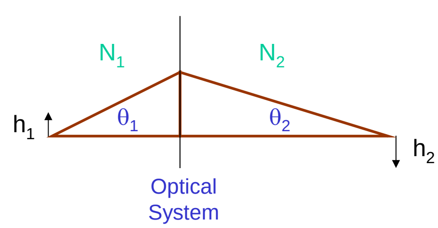
$$L_1(\theta_1/\theta_2)^2 = L_2$$

or: $\mathbf{L_1(N_2/N_1)^2 = L_2}$

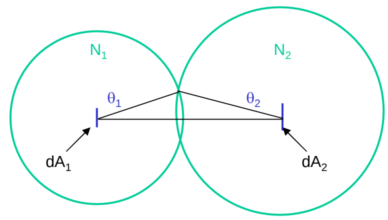
Abbe's Sine Condition: I

Suppose we have a source of height h_1 in medium of index N_1, imaged into medium N_2. We will show that the image height is related to the source height by:

$$\mathbf{N_1 h_1 \sin\theta_1 = N_2 h_2 \sin\theta_2}$$



Abbe's Sine Condition: II



Mirror walled blackbody enclosures at temperature, T.

Aperture images dA_1 onto dA_2

Thermodynamic Proof:

By the radiance theorem, if the radiance of dA_1, measured in the medium with index N_1 is $L_o(N_1)^2$, then the radiance of dA_2 measured into its surrounding medium must be $L_o(N_2)^2$.

Now, dA_1 radiates a power $d^2\Phi$ into an annular element of solid angle with half angle α of: $d^2\Phi = 2\pi \underbrace{L_o(N_1)^2}_{\text{Radiance theorem}/\Omega} dA_1 \cos\alpha \sin\alpha \, d\alpha$

Abbe's Sine Condition: III

Therefore, the total power transferred from dA_1 to dA_2 is:

$$d\Phi_1 = \int_0^{\theta_1} d^2\Phi_1 = \pi L_o(N_1)^2 dA_1 \sin^2\theta_1$$

where θ_1 is the half angle subtended by the aperture from the point of view of dA_1. Similarly, the power transferred from dA_1 to dA_2 is given by:

$$d\Phi_2 = \int_0^{\theta_2} d^2\Phi_2 = \pi L_o(N_2)^2 dA_2 \sin^2\theta_2$$

By the second law of thermodynamics: $d\Phi_1 = d\Phi_2$

$\Rightarrow$ $N_1 h_1 \sin\theta_1 = N_2 h_2 \sin\theta_2$

where h_1 and h_2 are the linear dimensions of A_1 and A_2

Étendue: I

In general, the *basic radiance, defined by: L/N^2* of a narrow beam of radiation *is conserved* as the beam propagates through any loss-less optical system. (see Boyd, section 5.2)

Total Power Measurement (Boyd, section 5.5)

What is the total power transmitted by a perfectly transmitting optical system (i.e. no vignetting, absorption, etc.)?

The power is given by:

$$\Phi = \int\int L(\mathbf{r},\mathbf{n})\, dA \cos\theta\, d\Omega$$

where $L(\mathbf{r},\mathbf{n})$ is the source radiance of the point $\mathbf{r}$ in the direction of unit vector $\mathbf{n}$. The surface integral is over the entrance window, and the solid angle integral extends over the solid angle subtended by the entrance window.

Étendue: II

To characterize the properties of the optical system assume the source is uniform and Lambertian ($L(\mathbf{n}) = L_o$), then:

$$\Phi = L_o \int\int dA \cos\theta\, d\Omega$$

$$= L_o/(N_o)^2\, \varepsilon$$

Lambertian source: one who's radiance is independent of the viewing angle.

where N_o = the index of refraction, and

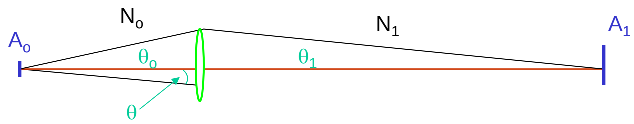
$$\varepsilon = \textbf{\textit{étendue}} \text{ of the system}$$

$$\equiv (N_o)^2 \int\int dA \cos\theta\, d\Omega$$

The étendue is a *purely geometric quantity* that is a measure of the flux gathering capability of the optical system. The collected power is the product of ε and the basic radiance of the source.

$$\text{power} = \underbrace{\text{étendue}}_{\text{area·solid angle}} \cdot \underbrace{\text{radiance}}_{\text{intensity (W/m}^2\text{/sr)}}$$

Étendue: III



Consider the optical system above. Suppose A_o, the area of the source is small so that the solid angle subtended by the entrance pupil of the optical system does not change over the source. Then :

$$\varepsilon = (N_o)^2 A_o \int \cos\theta \, d\Omega$$

or:

$$\varepsilon = (N_o)^2 A_o \Omega_{proj,\, o}$$

where we define the projected solid angle by:

$$\Omega_{proj,,o} = \int \cos\theta \, d\Omega$$

Étendue: IV

Since the entrance pupil subtends a half angle θ_o

$$\Omega_{proj,,o} = \int_0^{\theta_o} 2\pi \sin\theta \cos\theta \, d\theta$$
$$= \pi (\sin^2 \theta_o)$$

and hence:

$$\varepsilon = \pi (N_o)^2 A_o \sin^2 \theta_o$$

Now, recall that $Nh \sin\theta$ (h = height of the object/image) is conserved between the object and image in a well corrected imaging system (Abbe's sine condition). Therefore, the *étendue is invariant between the image and object planes.*

Étendue: V

More generally, consider an element of the étendue:

$$d^2\varepsilon = N^2\, dA \cos\theta\, d\Omega$$

If we also consider the flux passing through the same element of area into the same solid angle, and in the same direction:

$$d^2\Phi = L\, dA \cos\theta\, d\Omega$$

Which yields:

$$d^2\Phi = L/(N)^2 \cdot d^2\varepsilon$$

Now, in any loss-less system, by conservation of energy, $d^2\Phi$ is conserved. We also have by the general form of the radiance theorem, that $L/(N)^2$ is invariant. Hence, $d^2\varepsilon$ must be invariant. We have then, that:

$$\varepsilon = \iint d^2\varepsilon = N^2 \iint dA \cos\theta\, d\Omega$$

is conserved. The *étendue can be evaluated over any surface that intersects all the rays passing through the system.*

Étendue: VI

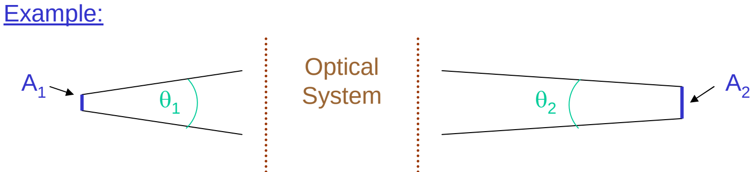
The invariance of étendue forms the basis for our usual statement that:

$A\Omega$ = constant

in an optical system. Keep in mind, however, that this expression is not the most general form.

The fact that $A\Omega$ is conserved provides a very powerful tool for optical system design.

Example:



$A\Omega$ = constant $\Rightarrow A_1\Omega_1 = A_2\Omega_2$ where $A = \pi/4\ d^2$, and $\Omega = \pi/4\ \theta^2$

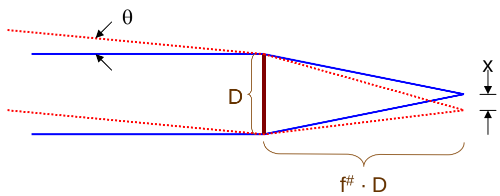
$$\Rightarrow d_1\,\theta_1 = d_2\,\theta_2$$

But, since $f\# = f/D \approx 1/\theta$

$$\Rightarrow d_1/f\#_1 = d_2/f\#_2$$

Plate Scale: I

Consider the case where we wish to match the image size from a telescope to the size of a pixel in a CCD camera.



For the Palomar telescope, with an f/15.7 secondary, the plate scale is given by:

$x_T \quad = f^{\#}_T \cdot D_T \cdot \theta_s$

$= 1''/206{,}265''/\text{radian x } 5000 \text{ mm x } 15.7$

$= 0.387$ mm for 1”

$\Rightarrow$ *plate scale* = 2.6”/mm

Thus, to cover 0.5” with a “pixel”, we need a detector with is 193 μm across.

Plate Scale: II

Typical CCD’s have pixels ~ 25 μm (x_d)across, so that we need to re-image to obtain the correct plate scale. From our relation AΩ = constant, we can easily determine the $f^{\#}_c$ of the final optical stage (camera).

$f^{\#}_c \quad = x_d/x_T \cdot f^{\#}_T$

$= 25/193 \cdot 15.7$

$= 2.0!!$ (a very fast camera!)

Note: $x_T \quad = \theta_s f^{\#}_T \cdot D_T$ Focal length, f

$\Rightarrow \quad f^{\#}_c \quad = x_d/(\theta_s D_T)$ 1/(plate scale)

i.e., we get the desired $f^{\#}$ of the final camera in terms of the plate scale desired, and the primary aperture.

Plate Scale: III

It is often useful to match the diffraction spot from the telescope to the size of the detector. This is relevant for diffraction limited (not seeing limited) observations. For diffraction from a filled aperture, the full width, at half maximum of the beam is given by:

$$\theta_{diff} = 1.22\ \lambda/D_T$$

Where D is the size of the primary mirror.

Therefore, since d = θ f#·D, we have

$$d_{diff} = 1.22\ \lambda/D_T \cdot f^{\#}_T \cdot D_T$$
$$= 1.22\ \lambda \cdot f^{\#}_T$$

"λ·f pixels"

For the visible, λ = 0.5 μm:

$$d_{diff} = 1.22 \cdot 0.5 \cdot 2.0$$
$$= 1.2\ \mu m$$

Therefore, the pixel size is large compared with the diffraction limit in this case.

Plate Scale: IV

At longer wavelengths, it is often possible to obtain the diffraction limited images. For example, the Hale 5 m telescope is diffraction limited at λ >10 μm, and approaches the diffraction limit for λ >2 μm if adaptive optics or speckle techniques are used. To obtain the diffraction limit in a single exposure, one needs to sample the focal plane at the *Nyquist frequency:* 2 pixels per diffraction limited beam:

$$\theta_{diff} = 1.22\ \lambda/D_T$$
$$\Rightarrow f^{\#}_c = d_{diff}/(\theta_{diff}/2 \cdot D_T)$$
$$\Rightarrow f^{\#}_c = d_{diff}/(1.22\ \lambda/2)$$

"λ·f over 2 pixels"

final f#

$$f^{\#}_c = d_{diff}/(0.61\lambda)$$

pixel size

Spectrocam-10 on the Hale 5 m telescope has 75 μm square pixels, and is designed to fully sample the focal plane at 10 μm. The final $f^{\#}$ must therefore be:

$$f^{\#}_c = 75\ \mu m/(0.61 \cdot 10\ \mu m) = 4.5$$