31. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

J	J	...
M	M	...
m_1	m_2	
m_1	m_2	Coefficients
.	.	
.	.	
.	.	

1/2×1/2

	+1		1	0
+1/2	+1/2	1	0	0
+1/2	−1/2	1/2	1/2	1
−1/2	+1/2	1/2	−1/2	−1
	−1/2	−1/2	1	

1×1/2

	+3/2		3/2	1/2
+1	+1/2	1	+1/2	+1/2
+1	−1/2	1/3	2/3	3/2	1/2
0	+1/2	2/3	−1/3	−1/2	−1/2
	0	−1/2	2/3	1/3	3/2
	−1	+1/2	1/3	−2/3	−3/2
		−1	−1/2	1	

2×1

	+3		3	2
+2	+1	1	+2	+2
+2	0	1/3	2/3	3	2	1
+1	+1	2/3	−1/3	+1	+1	+1
+2	−1	1/15	1/3	3/5	3	2	1
+1	0	8/15	1/6	−3/10	0	0	0
0	+1	6/15	−1/2	1/10	+1	−1	1/5	1/2	3/10
					0	0	3/5	0	−2/5	3	2	1
					−1	+1	1/5	−1/2	3/10	−1	−1	−1
										0	−1	6/15	1/2	1/10
										−1	0	8/15	−1/6	−3/10	3	2
										−2	+1	1/15	−1/3	3/5	−2	−2
															−1	−1	2/3	1/3	3
															−2	0	1/3	−2/3	−3
																−2	−1	1

1×1

	+2		2	1
+1	+1	1	+1	+1
+1	0	1/2	1/2	2	1	0
0	+1	1/2	−1/2	0	0	0
+1	−1	1/6	1/2	1/3	2	1
0	0	2/3	0	−1/3	−1	−1
−1	+1	1/6	−1/2	1/3	0	−1	1/2	1/2	2
					−1	0	1/2	−1/2	−2
						−1	−1	1	

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

2×1/2

	+5/2		5/2	3/2
+2	1/2	1	3/2	+3/2
+2	−1/2	1/5	4/5	5/2	3/2
+1	+1/2	4/5	−1/5	+1/2	+1/2
+1	−1/2	2/5	3/5	5/2	3/2
0	+1/2	3/5	−2/5	−1/2	−1/2
0	−1/2	3/5	2/5	5/2	3/2
−1	+1/2	2/5	−3/5	−3/2	−3/2
−1	−1/2	4/5	1/5	5/2
−2	+1/2	1/5	−4/5	−5/2
	−2	−1/2	1

3/2×1/2

	+2		2	1
+3/2	+1/2	1	+1	+1
+3/2	−1/2	1/4	3/4	2	1
+1/2	+1/2	3/4	−1/4	0	0
+1/2	−1/2	1/2	1/2	2	1
−1/2	+1/2	1/2	−1/2	−1	−1
−1/2	−1/2	3/4	1/4	2
−3/2	+1/2	1/4	−3/4	−2
−3/2	−1/2	1

3/2×1

	+5/2		5/2	3/2
+3/2	+1	1	+3/2	+3/2
+3/2	0	2/5	3/5	5/2	3/2	1/2
+1/2	+1	3/5	−2/5	+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2	5/2	3/2	1/2
+1/2	0	3/5	1/15	−1/3	−1/2	−1/2	−1/2
−1/2	+1	3/10	−8/15	1/6	+1/2	−1	3/10	8/15	1/6	5/2	3/2
					−1/2	0	3/5	−1/15	−1/3	−3/2	−3/2
					−3/2	+1	1/10	−2/5	1/2
										−1/2	−1	3/5	2/5	5/2
										−3/2	0	2/5	−3/5	−5/2
												−3/2	−1	1

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y^m_\ell\, e^{-im\phi}$$

$$d^{j}_{m',m} = (-1)^{m-m'} d^{j}_{m,m'} = d^{j}_{-m,-m'}$$

$$d^{1}_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1}_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^{1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^{1}_{1,-1} = \frac{1-\cos\theta}{2}$$

2×2

	+2 +2	+2 +1	+1 +2	+1 +1	+2 0	+1 1	0 2	+2 −2	+1 −1	0 0	−1 1	−2 2
4	1											
4		1/2	1/2									
3		1/2	−1/2									
4				3/14	1/2	2/7						
+2				4/7	0	−3/7						
+2				3/14	−1/2	2/7						
4							1/14	3/10	3/7	1/5		
3							3/7	1/5	−1/14	−3/10		
2							3/7	−1/5	−1/14	3/10		
1							1/14	−3/10	3/7	−1/5		
0											1/70	1/10	2/7	2/5	1/5
+2	−2										8/35	2/5	1/14	−1/10	−1/5
+1	−1										18/35	0	−2/7	0	1/5
0	0										8/35	−2/5	1/14	1/10	−1/5
−1	1										1/70	−1/10	2/7	−2/5	1/5
−2	2														

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^{2}_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{3,2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^{2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{3,2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^{2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2} \qquad d^{2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^{2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2} \qquad d^{2}_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2 \qquad d^{2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\,\cos\theta$$

$$d^{2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1) \qquad d^{2}_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

2×3/2

	+2 +3/2	+2 +1/2	+1 +3/2	+2 −1/2	+1 1/2	0 3/2	+1/2	+1/2	+1/2	+1/2	+2 −3/2	+1 −1/2	0 1/2	−1 3/2
7/2	1													
7/2		3/7	4/7											
5/2		4/7	−3/7											
+3/2				1/7	16/35	2/5								
+3/2				4/7	1/35	−2/5								
+3/2				2/7	−18/35	1/5								
7/2							1/35	6/35	2/5	2/5				
5/2							12/35	5/14	0	−3/10				
3/2							18/35	−3/35	−1/5	1/5				
1/2							4/35	−27/70	2/5	−1/10				
+1/2	−3/2										4/35	27/70	2/5	1/10
+1/2	−1/2										18/35	3/35	−1/5	−1/5
+1/2	+1/2										12/35	−5/14	0	3/10
+1/2	+3/2										1/35	−6/35	2/5	−2/5
5/2	−3/2													
3/2	−3/2													
3/2	−1/2													
3/2	+1/2													
1/2	−3/2													
1/2	−1/2													
1/2	+1/2													
1/2	+3/2													

3/2×3/2

	+3/2 +3/2	+3/2 +1/2	+1/2 +3/2	+3/2 −1/2	+1/2 +1/2	−1/2 +3/2	+3/2 −3/2	+1/2 −1/2	−1/2 +1/2	−3/2 +3/2	+1/2 −3/2	−1/2 −1/2	−3/2 +1/2	−1/2 −3/2	−3/2 −1/2	−3/2 −3/2
3	1															
3		1/2	1/2													
3		1/2	−1/2													
3				1/5	1/2	3/10										
2				3/5	0	−2/5										
1				1/5	−1/2	3/10										
3							1/20	1/4	9/20	1/4						
2							9/20	1/4	−1/20	−1/4						
1							9/20	−1/4	−1/20	1/4						
0							1/20	−1/4	9/20	−1/4						
3	−1										1/5	1/2	3/10			
2	−1										3/5	0	−2/5			
1	−1										1/5	−1/2	3/10			
3	−2													1/2	1/2	3
2	−2													1/2	−1/2	−3
1	−2															
3	−3															
2	−3															
1	−3															
4	−4															
3	−4															
2	−4															
1	−4															
0	−4															

3/2×2

	+2 +2	+2 +1	+1 +2	+1 +1	+2 0	+1 1	0 2	+2 −1	+1 0	0 1	−1 2	+1 −2	0 −1	−1 0	−2 1	0 −2	−1 −1	−2 0	−1 −2	−2 −1	−2 −2
4	1																				
4		1/2	1/2																		
3		1/2	−1/2																		
4				3/14	1/2	2/7															
+3				4/7	0	−3/7															
+3				3/14	−1/2	2/7															
4							1/14	3/10	3/7	1/5											
3							3/7	1/5	−1/14	−3/10											
2							3/7	−1/5	−1/14	3/10											
1							1/14	−3/10	3/7	−1/5											
0											1/70	1/10	2/7	2/5	1/5						
+2	−2										8/35	2/5	1/14	−1/10	−1/5						
+1	−1										18/35	0	−2/7	0	1/5						
0	0										8/35	−2/5	1/14	1/10	−1/5						
−1	1										1/70	−1/10	2/7	−2/5	1/5						
−2	2																				
4	−1															1/14	3/10	3/7	1/5		
3	−1															3/7	1/5	−1/14	−3/10		
2	−1															3/7	−1/5	−1/14	3/10		
1	−1															1/14	−3/10	3/7	−1/5		
0	−1																				
4	−2																			3/14	1/2	2/7
3	−2																			4/7	0	−3/7
2	−2																			3/14	−1/2	2/7
1	−2																					
0	−2																					
4	−3																					
3	−3																					
2	−3																					
1	−3																					
0	−3																					
4	−4																					
3	−4																					
2	−4																					
1	−4																					
0	−4																					