21-90. We are asked to calculate the electric field vector at a particular location $(x, 0)$ on the positive x-axis due to a line-of-charge that sits on the y-axis between the origin and the point $(0, a)$. Since the total charge on this line-of-charge is Q, the uniform linear charge density λ is Q/a. Begin by considering an infinitesimal segment (dy) of the line-of-charge located at $(0, y)$. The infinitesimal charge $dQ = \lambda dy$ contained in this infinitesimal segment produces an infinitesimal electric field vector with magnitude: $|d\vec{\mathbf{E}}| = k\,dQ/(x^2+y^2)$. $d\vec{\mathbf{E}}$ points into quadrant IV, i.e., its x component is positive $(dE_x > 0)$ and its y component is negative $(dE_y < 0)$. Using the angle θ we can say:

$$d\vec{\mathbf{E}} = \left(\frac{k\cos\theta\,dQ}{x^2+y^2}\,,-\,\frac{k\sin\theta\,dQ}{x^2+y^2}\right)$$

Geometry allows write:

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \qquad\qquad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

so:

$$d\vec{\mathbf{E}} = \left(\frac{kx\,dQ}{(x^2+y^2)^{3/2}}\,,-\,\frac{ky\,dQ}{(x^2+y^2)^{3/2}}\right)$$

Adding up (integrating) these $d\vec{\mathbf{E}}$, gives:

$$\begin{aligned}
\vec{\mathbf{E}} &= \left(\int_0^a \frac{k\lambda x}{(x^2+y^2)^{3/2}}\,dy\,,-\int_0^a \frac{k\lambda y}{(x^2+y^2)^{3/2}}\,dy\right)\\
&= \left(k\lambda x\int_0^a \frac{dy}{(x^2+y^2)^{3/2}}\,,-k\lambda\int_0^a \frac{ydy}{(x^2+y^2)^{3/2}}\right)\\
&= \left(k\lambda x\,\left.\frac{y}{x^2(x^2+y^2)^{1/2}}\right|_0^a\,,+k\lambda\,\left.\frac{1}{(x^2+y^2)^{1/2}}\right|_0^a\right)\\
&= \left(\frac{k\lambda a}{x(x^2+a^2)^{1/2}}\,,-k\lambda\left[\frac{1}{x}-\frac{1}{(x^2+a^2)^{1/2}}\right]\right)\\
&= \left(\frac{kQ}{x(x^2+a^2)^{1/2}}\,,-\,\frac{kQ}{ax}\left[1-(1+(a/x)^2)^{-1/2}\right]\right)\\
&\to \left(\frac{kQ}{x^2}\,,-\,\frac{kQ}{ax}\left[\frac{1}{2}(a/x)^2\right]\right) \quad \text{for: } x \gg a
\end{aligned}$$

where we have used the binomial approximation: $(1+\epsilon)^a \approx 1 + a\epsilon$ for $\epsilon \ll 1$ (see p. 435 in chapter 13).

Since the force on a charge $-q$ is $\vec{\mathbf{F}} = -q\vec{\mathbf{E}}$, the remaining parts of the problem follow immediately. Note that approximation of $\vec{\mathbf{E}}$ for $x \gg a$ is equivalent to the statement that the line-of-charge Q acts nearly as if it were a point charge located at $(0, a/2)$.

21-97. Note first that symmetry suggests that the electric field will point along the diagonal. We are asked to calculate the electric field vector at the origin due to a line-of-charge that sits on a circular arc of radius a. Since the total charge on this line-of-charge is $-Q$, the uniform linear charge density λ is $-2Q/\pi a$. Begin by considering an infinitesimal segment $(d\theta)$ of the line-of-charge located at θ. The infinitesimal charge $dQ = \lambda a d\theta$ contained in this infinitesimal segment produces an infinitesimal electric field vector with magnitude: $|d\vec{\mathbf{E}}| = k\,|dQ|/a^2$. $d\vec{\mathbf{E}}$ points into quadrant I at an angle θ (i.e., right at the source charge since the arc has a negative charge). Using the angle θ we can say:

$$\begin{aligned}
d\vec{\mathbf{E}} &= \left(\frac{k\cos\theta\,|dQ|}{a^2}\,,\frac{k\sin\theta\,|dQ|}{a^2}\right)\\
&= \frac{k|\lambda|}{a}\,(\cos\theta\,d\theta\,,\sin\theta\,d\theta)
\end{aligned}$$

Adding up (integrating) these $d\vec{\mathbf{E}}$, gives:

$$\begin{aligned}
\vec{\mathbf{E}} &= \frac{k|\lambda|}{a}\left(\int_0^{\pi/2}\cos\theta\,d\theta\,,\,\int_0^{\pi/2}\sin\theta\,d\theta\right)\\
&= \frac{k|\lambda|}{a}\left(\sin\theta\Big|_0^{\pi/2}\,,\,-\cos\theta\Big|_0^{\pi/2}\right)\\
&= \frac{k|\lambda|}{a}\,(1,1)\\
&= \frac{k2|Q|}{\pi a^2}\,(1,1)
\end{aligned}$$

old exam: #14—I assume λ and d are positive, so we are being asked to find $\vec{\mathbf{E}}$ someplace on the negative x axis; clearly the direction of every $d\vec{\mathbf{E}}$ due to every bit of charge on the line-of-charge will be away from the charge, i.e., in the direction $-\hat{\mathbf{i}}$. Since all the $d\vec{\mathbf{E}}$ are in the same direction we can add them up just like real numbers. For the remainder of the solution we focus on the magnitude of this vector: $|\vec{\mathbf{E}}| \equiv E$. Begin by considering an infinitesimal segment (dx) of the line-of-charge located at $(x, 0)$. The infinitesimal charge $dQ = \lambda\,dx$ contained in this infinitesimal segment produces an infinitesimal electric field vector with magnitude: $|d\vec{\mathbf{E}}| = dE = k\,dQ/(x+d)^2$. Adding up (integrating) these dE, gives:

$$\begin{aligned}
E &= \int_0^L \frac{k\lambda\,dx}{(x+d)^2}\\
&= k\lambda\int_0^L \frac{dx}{(x+d)^2}\\
&= k\lambda\left(\frac{(x+d)^{-1}}{-1}\Bigg|_0^L\right)\\
&= k\lambda\left(\frac{1}{d}-\frac{1}{L+d}\right)\\
&= \frac{k\lambda}{d}\left(1-(1+L/d)^{-1}\right)\\
&\to \frac{k\lambda L}{d^2}\ \text{for: } d \gg L
\end{aligned}$$

where we have again used the binomial approximation: $(1+\epsilon)^a \approx 1 + a\epsilon$ for $\epsilon \ll 1$ (see p. 435 in chapter 13). Do note: $\vec{\mathbf{E}} = -E\,\hat{\mathbf{i}}$.

Note that approximation of E for $d \gg L$ is equivalent to the statement that the line-of-charge which has total charge $Q = \lambda L$ acts nearly as if it were a point charge located at $(0, 0)$.