23-66. Calculate the potential due to each thin ring and integrate over the disk to find the potential. This problem is similar to Example 21.12, except since V is a scalar no components are involved. Divide the disk into rings of radius r; each ring has area $dA = 2\pi r\ dr$ and hence charge $dQ = \sigma\ 2\pi r\ dr$. On the x axis, every bit of that ring is at a distance $\sqrt{r^2 + x^2}$, and so the corresponding electric potential due to the ring is:

$$dV = \frac{kdQ}{\sqrt{r^2+x^2}} = \frac{\sigma\ 2\pi r\ dr}{4\pi\epsilon_0\sqrt{r^2+x^2}} = \frac{\sigma}{2\epsilon_0}\ \frac{r\ dr}{\sqrt{r^2+x^2}}$$

Adding up the potential due to all the rings that make up the disk results in an integral:

$$V = \frac{\sigma}{2\epsilon_0}\int_0^R \frac{r\ dr}{\sqrt{r^2+x^2}}$$

This integral can be simplified by using the substitution: $u = r^2 + x^2$ and $du = 2r\ dr$:

$$V = \frac{\sigma}{2\epsilon_0}\int_{x^2}^{R^2+x^2} \frac{\frac{1}{2}\ du}{u^{1/2}} = \frac{\sigma}{4\epsilon_0}\left[\frac{u^{1/2}}{\frac{1}{2}}\right]_{x^2}^{R^2+x^2} = \frac{\sigma}{2\epsilon_0}\ \left[(R^2+x^2)^{1/2} - |x|\right]$$

Note in passing that for $x \gg R$ this expression approximates the potential of a point charge of magnitude $Q = \sigma\pi R^2$ at the origin:

$$V = \frac{\sigma}{2\epsilon_0}\ |x|\left[(1+(R/x)^2)^{1/2} - 1\right] \to \frac{\sigma}{2\epsilon_0}\ |x|\left[\frac{1}{2}(R/x)^2\right] = \frac{\sigma\pi R^2}{4\pi\epsilon_0|x|}$$

Assuming $x > 0$ (so $|x| = x$) we have for the electric field:

$$E_x = -\frac{\partial V}{\partial x} = \frac{\sigma}{2\epsilon_0}\frac{\partial}{\partial x}\left(x - (R^2+x^2)^{1/2}\right) = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{x}{(R^2+x^2)^{1/2}}\right)$$

23-69. We use Eq. 23-18: $V_a - V_b = -\int_b^a \vec{\mathbf{E}}\cdot d\vec{\ell}$. We select the path of integration to be the x axis from the point $P = (x, 0)$ to ∞. Thus $d\vec{\ell} = dx\ \hat{\mathbf{i}}$, $a = \infty$, $b = x$, $V_a = 0$ so we have:

$$V_b = \int_x^\infty E_x\ dx = \frac{Q}{4\pi\epsilon_0}\int_x^\infty \frac{x}{(x^2+a^2)^{3/2}}\ dx$$

This integral can be simplified by using the substitution: $u = x^2 + a^2$ and $du = 2x\ dx$:

$$V_b = \frac{Q}{4\pi\epsilon_0}\int_{x^2+a^2}^\infty \frac{\frac{1}{2}du}{u^{3/2}} = \frac{Q}{8\pi\epsilon_0}\left[\frac{u^{-1/2}}{-\frac{1}{2}}\right]_{x^2+a^2}^\infty = \frac{Q}{4\pi\epsilon_0}\ \left[(x^2+a^2)^{-1/2}\right] = \frac{Q}{4\pi\epsilon_0\sqrt{x^2+a^2}}$$

Please forgive a double meaning for x: both as an end-point for the region of integration and as the dummy variable of integration.

23-79. For this same charge configuration we have previously found the electric field on the x-axis (old exam #14) and [equivalent to] on the y axis: 21-90. We also used these result in problem 21-95.

(a) It will be convenient to call the x location of point P "d" so that x can be unambiguously used to refer to the location of the source charge: $-a < x < 0$. The linear charge density of the wire can be denoted: $\lambda = Q/a$. If we consider some infinitesimal segment dx of the line-of-charge, the distance between that segment and the point P is $d-x$. (Note that this is the same as $d+|x|$, since $x<0$.) Since the charge of this segment is $dQ=\lambda\ dx$, the resulting electric potential due to the segment dx is:

$$dV=\frac{k\ dQ}{d-x}=\frac{\lambda\ dx}{4\pi\epsilon_0(d-x)}$$

Adding up the potential due to all the segments that make up the line-of-charge results in an integral:

$$V=\frac{\lambda}{4\pi\epsilon_0}\int_{-a}^{0}\frac{dx}{d-x}=\frac{\lambda}{4\pi\epsilon_0}\Big[-\ln(d-x)\Big]_{-a}^{0}=\frac{\lambda}{4\pi\epsilon_0}\left[\ln(d+a)-\ln(d)\right]=\frac{\lambda\ln(1+a/d)}{4\pi\epsilon_0}$$

Substituting back $d\to x$ and $\lambda\to Q/a$ yields:

$$V=\frac{Q\ln(1+a/x)}{4\pi\epsilon_0 a}$$

Since $\ln(1+\epsilon)\approx\epsilon$

$$V\approx\frac{Q}{4\pi\epsilon_0 x}\qquad\text{for: } x\gg a$$

which is equivalent to the potential of a point charge Q at the origin.

(b) If we consider some infinitesimal segment dx of the line of charge, the distance between that segment and the point R (a distance y up the y axis) is $\sqrt{x^2+y^2}$. Since the charge of this segment is $dQ=\lambda\ dx$, the resulting electric potential due to the segment dx is:

$$dV=\frac{k\ dQ}{\sqrt{x^2+y^2}}=\frac{\lambda\ dx}{4\pi\epsilon_0\sqrt{x^2+y^2}}$$

Adding up the potential due to all the segments that make up the line-of-charge results in an integral:

$$V=\frac{\lambda}{4\pi\epsilon_0}\int_{-a}^{0}\frac{dx}{\sqrt{x^2+y^2}}=\frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(x+\sqrt{x^2+y^2}\right)\right]_{-a}^{0}=\frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(\frac{y}{-a+\sqrt{a^2+y^2}}\right)\right]$$

If you want, you can make this answer agree with the book by "simplification":

$$\frac{y}{-a+\sqrt{a^2+y^2}}\cdot\frac{\sqrt{a^2+y^2}+a}{\sqrt{a^2+y^2}+a}=\frac{y(\sqrt{a^2+y^2}+a)}{(a^2+y^2)-a^2}=\frac{\sqrt{a^2+y^2}+a}{y}=\sqrt{1+(a/y)^2}+(a/y)$$

Using either form, if $y\gg a$, then $(a/y)\gg(a/y)^2$ and we approximately have:

$$V\approx\frac{\lambda}{4\pi\epsilon_0}\ \ln\left(1+(a/y)\right)\approx\frac{\lambda a}{4\pi\epsilon_0 y}=\frac{Q}{4\pi\epsilon_0 y}$$

i.e., like a point charge Q at the origin.