28-69. For the straight horizontal and vertical segments of the wire $d\vec{\boldsymbol{\ell}} \parallel \vec{\mathbf{r}}$ and so the cross product is zero. (Technically for the vertical segment $d\vec{\boldsymbol{\ell}} \parallel \vec{\mathbf{r}}$, i.e., $\phi = 0$, whereas for the horizontal segment $d\vec{\boldsymbol{\ell}}$ and $\vec{\mathbf{r}}$ are anti-parallel, i.e., $\phi = 180°$, but both parallel and anti-parallel produce zero cross product.) For any point on the arc, $d\vec{\boldsymbol{\ell}} \perp \vec{\mathbf{r}}$, and $d\vec{\boldsymbol{\ell}} \times \vec{\mathbf{r}}$ is out-of-page. Since all $d\vec{\mathbf{B}}$ are in the same direction we can just add them up as real numbers, with result:

$$B = \frac{\mu_0 \ell I}{4\pi R^2} = \frac{\mu_0 \frac{1}{2}\pi R I}{4\pi R^2} = \frac{\mu_0 I}{8R}$$

This process can be made unnecessarily explicit by specifying all the vector components involved in the arc (as we did in class). I define the vector $\vec{\boldsymbol{\ell}}$ to point from the origin to bits of the wire arc, and then $d\vec{\boldsymbol{\ell}}$ can be an infinitesimal step around the wire arc:

$$\begin{aligned}
\vec{\boldsymbol{\ell}} &= R\cos\theta\,\hat{\mathbf{i}} + R\sin\theta\,\hat{\mathbf{j}} \\
d\vec{\boldsymbol{\ell}} &= -Rd\theta\sin\theta\,\hat{\mathbf{i}} + Rd\theta\cos\theta\,\hat{\mathbf{j}}
\end{aligned}$$

where $\theta \in (\pi/2, \pi)$. $\vec{\mathbf{r}}$ points from the wire arc to the point-of-observation (the origin) and hence $\vec{\mathbf{r}} = -\vec{\boldsymbol{\ell}}$.

$$\vec{\mathbf{r}} = -R\cos\theta\,\hat{\mathbf{i}} - R\sin\theta\,\hat{\mathbf{j}}$$

We can now calculate the cross product of $d\vec{\boldsymbol{\ell}}$ and $\vec{\mathbf{r}}$:

$$d\vec{\boldsymbol{\ell}} \times \vec{\mathbf{r}} = R^2 d\theta \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\sin\theta & \cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \end{vmatrix} = R^2 d\theta\,(\sin^2\theta + \cos^2\theta)\hat{\mathbf{k}} = R^2 d\theta \hat{\mathbf{k}}$$

and then:

$$\vec{\mathbf{B}} = \int \frac{\mu_0 I d\vec{\boldsymbol{\ell}} \times \vec{\mathbf{r}}}{4\pi r^3} = \mu_0 I \int_{\pi/2}^{\pi} \frac{d\theta\,\hat{\mathbf{k}}}{4\pi R} = \frac{\mu_0 I}{8R}\,\hat{\mathbf{k}}$$

current sheet: Since $r = \sqrt{x^2 + d^2}$ we have:

$$B_x = \frac{\mu_0 K d}{2\pi} \int_{-\infty}^{+\infty} \frac{dx}{x^2 + d^2} = \frac{\mu_0 K d}{2\pi}\,\frac{1}{d}\tan^{-1}\left(\frac{x}{d}\right)\Bigg|_{-\infty}^{+\infty} = \frac{\mu_0 K}{2\pi}\,\pi = \frac{\mu_0 K}{2}$$

old exam #11 The distances the wires are from the point of observation are (respectively) $r_1 = \sqrt{s^2 - (s/2)^2} = \sqrt{3}s/2 = 0.0866$ m, $r_2 = r_3 = .05$ m. The right hand rule gives us the direction of the resulting vectors;

$$\begin{aligned}
\vec{\mathbf{B}}_1 &= \frac{\mu_0 I_1}{2\pi r_1}\,\hat{\mathbf{i}} \\
\vec{\mathbf{B}}_2 &= \frac{\mu_0 I_2}{2\pi r_2}\,\hat{\mathbf{j}} \\
\vec{\mathbf{B}}_3 &= \frac{\mu_0 I_3}{2\pi r_3}\,\hat{\mathbf{j}}
\end{aligned}$$

The net magnetic field is then:

$$\vec{\mathbf{B}} = \frac{\mu_0}{2\pi}\Big((I_1/r_1)\,\hat{\mathbf{i}} + (I_2/r_2 + I_3/r_3)\,\hat{\mathbf{j}}\Big) = \Big(.231\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}}\Big) \times 10^{-5}\ \mathrm{T}$$

old exam #14

A. The magnetic field is in the $-\hat{\mathbf{k}}$ direction.

$$B = \frac{\mu_0 N I R^2}{2\left(z^2+R^2\right)^{3/2}} = 2.85 \times 10^{-5}\ \mathrm{T}$$

B. The magnetic dipole is in the $-\hat{\mathbf{k}}$ direction.

$$\mu = NIA = 25 \cdot 2 \cdot \pi 0.15^2 = 3.53\ \mathrm{A \cdot m^2}$$

C.

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\mathbf{B}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -3.53 \\ 2 & 3 & 4 \end{vmatrix} = 3 \cdot 3.53\ \cdot \hat{\mathbf{i}} - 2 \cdot 3.53\ \cdot \hat{\mathbf{j}} = \left(10.6\,\hat{\mathbf{i}} - 7.07\,\hat{\mathbf{j}}\right)\ \mathrm{N \cdot m}$$