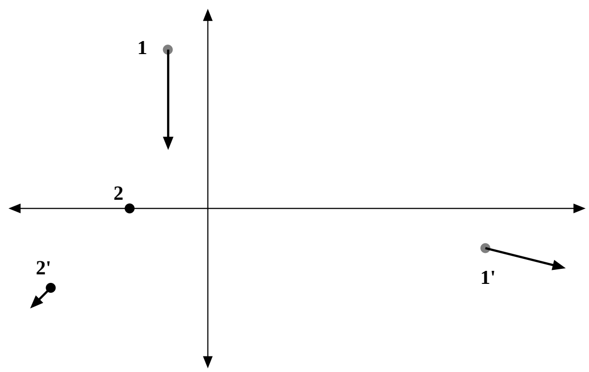
Complete problems four of the below problems

1. Particles 1 and 2 have a short-range, conservative, central force acting between them with no external forces present. Particle 1, with mass $m_1 = 1$ kg, moves straight down and "collides" (interacts via the short-range force) with particle 2 (which has mass $m_2 = 4$ kg). The below lists a pre-collision (unprimed) and a post-collision (primed) position (in m) and velocity (in m/s). Note: the prime denotes post-collision not CM-relative.

particle	pre-collision		post-collision	
1	$\mathbf{r}_1 = -\mathbf{i} + 4\mathbf{j}$	$\mathbf{v}_1 = -4\mathbf{j}$	$\mathbf{r}'_1 = 7\mathbf{i} - \mathbf{j}$	$\mathbf{v}'_1 = \frac{16}{5}\,\mathbf{i} - \frac{4}{5}\,\mathbf{j}$
2	$\mathbf{r}_2 = -2\mathbf{i}$	$\mathbf{v}_2 = \mathbf{0}$	$\mathbf{r}'_2 = -4\mathbf{i} - 2\mathbf{j}$	$\mathbf{v}'_2 = -\frac{4}{5}\,\mathbf{i} - \frac{4}{5}\,\mathbf{j}$



   (a) In the pre-collision state, find the location and velocity of the center of mass, $\mathbf{R}_{\text{cm}}, \mathbf{V}_{\text{cm}}$. Find the velocity of the center of mass in the post-collision state: $\mathbf{V}'_{\text{cm}}$. Is total momentum conserved?

   (b) In the pre-collision state, find the total angular momentums, $\mathbf{L}_{\text{total}}$. In the post-collision state, find the orbital angular momentum, $\mathbf{L}'_{\text{orbit}}$ (aka, angular momentum "OF" the CM) and the spin angular momentum, $\mathbf{L}'_{\text{spin}}$ (aka, angular momentum "ABOUT" the CM). (FYI: I think the easiest way to find $\mathbf{L}'_{\text{spin}}$ involves reduced mass.) Is total angular momentum conserved?

   (c) The relative velocity $(\mathbf{v}_1 - \mathbf{v}_2)$ in the pre-collision state is $-4\mathbf{j}$ and in the post-collision state it's $4\mathbf{i}$. Why does this allow us to conclude that kinetic energy is conserved?

2. Consider a point (mass m) that moves as if attached to the outer edge of a disk (with radius R) rolling upright on a level plane at a constant speed v. It can be shown that the position of the point is given by the vector:

$$\mathbf{r}(t) = (vt - R\sin(vt/R))\,\mathbf{i} + R(1 - \cos(vt/R))\,\mathbf{j}$$

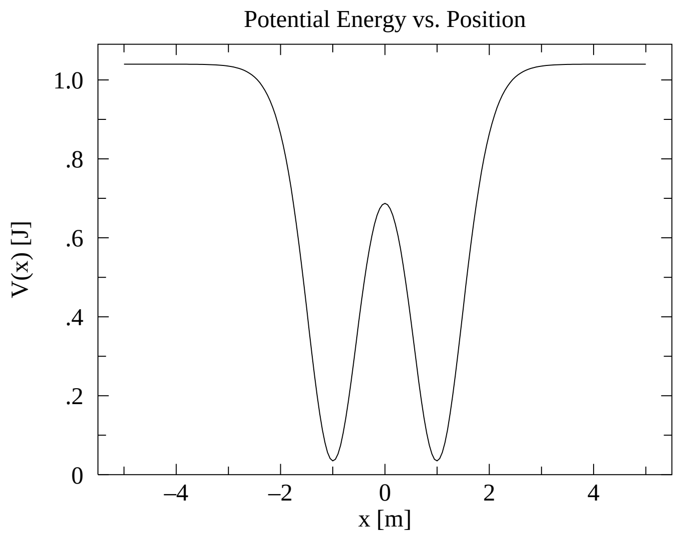
At $t = 0$ the point is at the bottom of the disk ($\mathbf{r}(0) = \mathbf{0}$); at time $vt = \pi R$ the point is at the top of the disk: $\mathbf{r} = \pi R\,\mathbf{i} + 2R\,\mathbf{j}$. (Note $\mathbf{j}$ is the vertical direction and $\mathbf{i}$ is in the direction of travel.)

   (a) Find the velocity and acceleration of the point at time t.

   (b) Use your result to calculate the velocity and acceleration of the point when it is at the top and bottom of the disk.

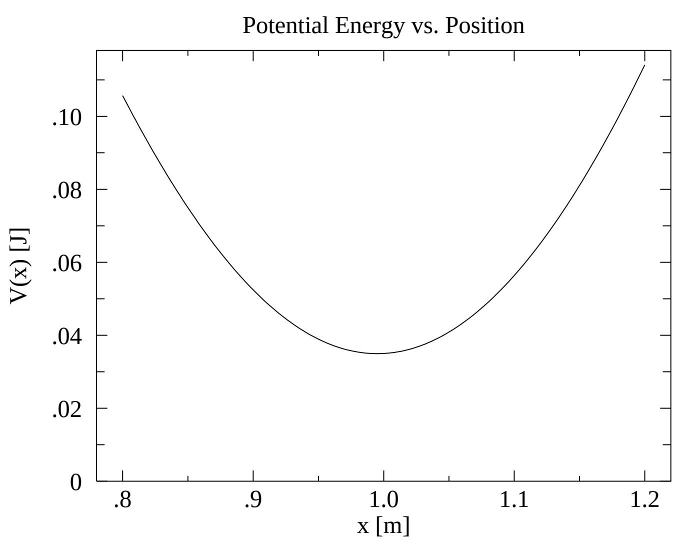
   (c) (Yes/No answers; no calculation required) Is there a net force on the particle when it is at the bottom? when it is at the top? Is there a net torque on the particle when it is at the bottom? when it is at the top?

3. The following plots display the potential energy (in J) of a particular force as a function of x measured in meters. The second plot displays a detail near $x = 1$ of the first.



(a) Report: an x value that is a stable equilibrium point, an x value that is an unstable equilibrium point, an x value for which the force pushes in the positive x direction, and an x value for which the force pushes in the negative x direction. The potential energy plot is quite flat for $|x| > 5$, but remains at a value a bit above 1 J. What can you conclude about the force in the region $|x| > 5$?

(b) Describe the future trajectory of a particle released at $x = 1$ with a total energy of 0.5 J. Describe the future trajectory of a particle released at $x = 1$ with a total energy of 1.0 J. Describe the future trajectory of a particle released at $x = 1$ with a total energy of 1.5 J.



(c) Estimate (numerically in Newtons) the force near $x = 1.15$.

4. Consider the linear *first-order* homogeneous differential equation (A) and its related inhomogeneous differential equation (B) (τ is a constant):

$$\tau \frac{dx}{dt} + x = 0 \qquad \text{(A)}$$
$$\tau \frac{dx}{dt} + x = F(t) \qquad \text{(B)}$$

(a) What is the solution to the homogeneous differential equation (A)?

(b) If $F(t) = f_0\, e^{i\omega t}$ then there is a solution to (B) of the form: $x(t) = \mathcal{A}e^{i\omega t} = Ae^{i(\omega t-\delta)}$, where $\mathcal{A} = Ae^{-i\delta}$ (i.e., $\mathcal{A} \in \mathbb{C}$ is a complex number) and A and δ are real numbers ($\mathbb{R}$) that depend on ω. Find this solution and report what A and δ are as (real-valued) functions of ω.

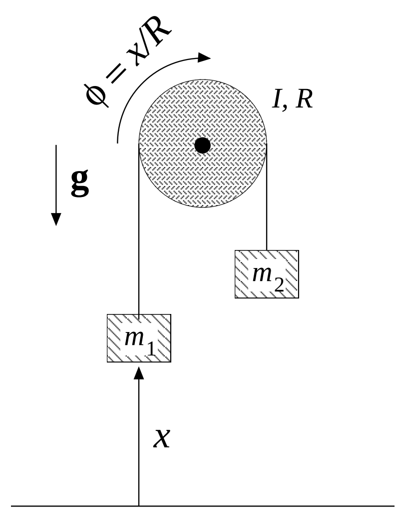
(c) A square wave $F(t)$ has a Fourier expansion:

$$F(t) = \cos(\omega t) - \frac{1}{3}\cos(3\omega t) + \frac{1}{5}\cos(5\omega t) + \cdots$$

Carefully write down the first two terms of the sum that describes the response $x(t)$ to this square-wave driving force. Your answer should involve the functions $A()$ and $\delta()$ you defined above evaluated at the appropriate frequencies.

5. In Atwood's Machine two masses (m_1 & m_2), connected by a string, hang off opposite ends of a frictionless pulley (radius R; moment of inertia I). If m_1 moves up a distance x the pulley turns an angle $\phi = x/R$ (why?) and the mass m_2 falls a distance x. In homework you showed that the acceleration ($\ddot{x}$) was given by:

$$\ddot{x} = \frac{(m_2 - m_1)g}{m_1 + m_2 + I/R^2}$$



(a) Write down the kinetic energy of the entire system in terms $\dot{x}$.

(b) Write down the potential energy of the entire system in terms of x.

(c) Since gravity is a conservative force the above energy should be a constant and hence its time derivative should be zero. Take the time derivative of your total energy and derive the above formula for the acceleration $\ddot{x}$.