Hamilton's Revenge

As described in the textbook, Euler Angles are a way to specify the configuration of a 3d object. Starting from a fixed configuration the desired configuration is obtained by a three step process:

1. rotation about the z axis by an angle ϕ
2. rotation about the y' axis[1] (i.e., the rotated y axis) by an angle θ
3. rotation about the z'' axis (i.e., the doubly rotated z axis which, in the end, is the body axis 3) by an angle ψ

I strongly recommend looking at the Wiki visualization (`Euler2a.gif`, author Juansempere — note it uses an alternative angle sequence) and video (https://youtu.be/N7AVc5yYX-k author Yudintsev or class web site `323_rotation_sequence_Euler.mp4` — note it swaps ψ and ϕ). I hope it is clear that almost certainly the object did not achieve its configuration by these three rotations — just as it's unlikely that an object reached a particular position by successive motions in the x, y and z directions. We are recording configuration not history.

The body-fixed frame (123), with aligned principal axes, is most convenient for calculation. However, we often need to know what a body-fixed vector looks like in the inertial frame (xyz). We define matrices to reverse the above three steps:

$$\mathcal{M}_\phi = \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

$$\mathcal{M}_\theta = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \tag{2}$$

$$\mathcal{M}_\psi = \begin{pmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

where:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathcal{M}_\phi \mathcal{M}_\theta \mathcal{M}_\psi \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{4}$$

Note: To make the reverse transformation (i.e., $(x, y, z) \to (x_1, x_2, x_3)$) you would apply the inverse matrices in the reverse order to (x, y, z). The inverse matrices are easily generated by negating the angle (e.g., $\theta \to -\theta$) or taking the matrix transpose.

We begin by using *Mathematica* to find the relation between $\dot\phi, \dot\theta, \dot\psi$ and $\boldsymbol{\omega}$ (in the body-fixed frame).

[1] This is the convention of our textbook. Often — e.g., Goldstein's *Classical Mechanics* and Wiki — the second rotation is made about the x' axis. Warning: before you lift a formula from a textbook you need to know which standard was used.

```
{0,0,dpsi}+Inverse[mpsi].{0,dtheta,0}+Inverse[mpsi].Inverse[mtheta].{0,0,dphi}
Simplify[%]
w=%
Out[7]= {dtheta Sin[psi] - dphi Cos[psi] Sin[theta],

>     dtheta Cos[psi] + dphi Sin[psi] Sin[theta], dpsi + dphi Cos[theta]}
```

$$\boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ \dot\psi \end{pmatrix} + \mathcal{M}_\psi^{-1} \begin{pmatrix} 0 \\ \dot\theta \\ 0 \end{pmatrix} + \mathcal{M}_\psi^{-1}\mathcal{M}_\theta^{-1} \begin{pmatrix} 0 \\ 0 \\ \dot\phi \end{pmatrix} \tag{5}$$

$$= \left(\dot\theta \sin(\psi) - \dot\phi\cos(\psi)\sin(\theta),\ \dot\theta\cos(\psi) + \dot\phi\sin(\psi)\sin(\theta),\ \dot\phi\cos(\theta) + \dot\psi\right) \tag{6}$$

Given $\boldsymbol{\omega}$ in the body-fixed frame it's easy (for *Mathematica*) to calculate the kinetic energy:

$$T = \frac{1}{2}\ \boldsymbol{\omega} \cdot \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \cdot \boldsymbol{\omega} \tag{7}$$

$$= \frac{1}{2}\ I_1 \left(\dot\phi^2 \sin^2(\theta) + \dot\theta^2\right) + \frac{1}{2}\ I_3 \left(\dot\phi\cos(\theta) + \dot\psi\right)^2 \tag{8}$$

The problem at hand is *free* precession... no external forces or potential energy; the Lagrangian is just the kinetic energy T. Notice that ϕ and ψ are cyclic (a.k.a., ignorable) coordinates so the corresponding canonical (a.k.a., generalized) momenta are constants:

$$p_\psi = \frac{\partial T}{\partial \dot\psi} = I_3\left(\dot\phi\cos(\theta) + \dot\psi\right) \tag{9}$$

$$p_\phi = \frac{\partial T}{\partial \dot\phi} = I_3\cos(\theta)\left(\dot\phi\cos(\theta) + \dot\psi\right) + I_1\ \dot\phi\sin^2(\theta) = p_\psi\cos(\theta) + I_1\ \dot\phi\sin^2(\theta) \tag{10}$$

Comparing to Eq. (6), see that $p_\psi = L_3$ (i.e., the angular momentum in the body-fixed z direction); at the end of this document we discovery $p_\phi = L_z$ (i.e., the angular momentum in the inertial frame z direction). Using these (constant) momenta we can rewrite the kinetic energy much as in a Hamiltonian (currently we leave $\dot\theta$ alone):

$$T = \frac{1}{2}\ I_1\ \dot\theta^2 + \frac{(p_\phi - p_\psi\cos(\theta))^2}{2I_1\sin^2(\theta)} + \frac{p_\psi^2}{2I_3} = \frac{1}{2}\ I_1\ \dot\theta^2 + V(\theta)$$

This expression now just involves constants and θ and $\dot\theta$; furthermore it is itself a constant. The usual logic of 1d conservation of energy applies to θ: turning points, equilibrium points, etc. In particular the minimum of $V(\theta)$ must be a stable equilibrium point where $\dot\theta = 0, \theta = \theta_0$ is a valid 'motion'. Working in terms of $c = \cos\theta$ note:

$$V(c) \propto \frac{(p_\phi - p_\psi c)^2}{1 - c^2} + \text{constant}$$

and $V' = 0$ has two solutions: $c = p_\phi/p_\psi$ and $c = p_\psi/p_\phi$. From Eq. (10), see that the first solution results in $\dot\phi = 0$ in addition to $\dot\theta = 0$. Applying those results to $\boldsymbol{\omega}$ see that $\boldsymbol{\omega}$ (and hence $\mathbf{L}$) is entirely along the body-fixed 3 axis. This is an object spinning in space with no additional motion; $c = p_\phi/p_\psi$ is just the static triangle of $\mathbf{L}$. The kinetic energy is simply: $p_\psi^2/(2I_3)$—the kinetic energy of rotation just about the body-fixed 3 axis.

The second solution is more interesting. Using the constant values of $p_\psi, p_\phi, \cos\theta$ find the values of $\dot\phi$ and $\dot\psi$:

```
Solve[{Pphi==Ppsi Ppsi/Pphi + dphi I1 (1- (Ppsi/Pphi)^2),
Ppsi== I3 (dpsi + dphi (Ppsi/Pphi))},{dpsi,dphi}]

                     (I1 - I3) Ppsi          Pphi
Out[19]= {{dpsi -> --------------, dphi -> ----}}
                         I1 I3                I1
```

Thus a free body moving with

$$\theta = \theta_0 \tag{11}$$

$$\phi = \dot\phi_0\, t = \frac{p_\psi}{I_1\cos\theta_0}\, t = \frac{I_3\omega_3}{I_1\cos\theta_0}\, t \tag{12}$$

$$\psi = \frac{I_1 - I_3}{I_1}\,\frac{p_\psi}{I_3}\, t = \frac{I_1 - I_3}{I_1}\,\omega_3\, t \equiv \Omega_b\, t \tag{13}$$

solves the equations of motion. Note that (θ, ϕ) define the direction of the body-fixed 3 axis in the inertial frame; evidently it is inclined (at θ_0) and rotating at rate $\dot\phi_0$. Using this solution we can calculate $\boldsymbol{\omega}$ in the body frame:

```
w /. {dphi->Ppsi/I1/Cos[theta], dpsi->(I1/I3-1)Ppsi/I1,dtheta->0}
Simplify[%]
```

$$\boldsymbol{\omega} = \left(-\frac{p_\psi\tan\theta_0}{I_1}\,\cos\psi,\; \frac{p_\psi\tan\theta_0}{I_1}\,\sin\psi,\; \frac{p_\psi}{I_3}\right) \tag{14}$$

i.e., ω_3 has a constant value of p_ψ/I_3 while $\boldsymbol{\omega}_\perp$ is rotating at rate Ω_b and has constant magnitude $p_\psi\tan\theta_0/I_1$. (If $I_1 > I_3$ this is a *clockwise* rotation about the 3 axis.) Since the moment of inertia tensor is diagonal in the body frame, it should be clear $\mathbf{L}$ is rotating in step with $\boldsymbol{\omega}$ at an angle θ_0 from the 3 axis. If $I_1 > I_3$ (prolate; pencil-like) $\boldsymbol{\omega}$ circles inside of $\mathbf{L}$s circling of the 3 axis. If $I_1 < I_3$ (oblate; pancake-like) $\mathbf{L}$ circles inside of $\boldsymbol{\omega}$s circling of the 3 axis. Since the body is stationary in the body-fixed frame it may sound odd to talk about the spin vector in that frame. Nevertheless that is exactly what we mean when we, standing on the Earth, refer to the Earth's spin axis. And in fact the Earth's spin axis is making a small loop around the 'north pole' (i.e., the Earth's symmetry axis): the Chandler wobble with $\theta_0 \sim 0.2$" and a period of about 433 days.

If we transform $\mathbf{L}$ from the body-fixed frame back into the inertial frame and substitute in the now known values for $\dot\psi, \dot\phi$ and $\dot\theta = 0$.

```
mphi.mtheta.mpsi.L
Simplify[%]
% /. {dphi->Pphi/I1, dpsi->Cos[theta](I1/I3-1)Pphi/I1,dtheta->0}
Simplify[%]

Out[24]= {0, 0, Pphi}
```

We conclude that this solution has **L** in the inertial frame aligned with the z axis. Note the consequence: $\dot{\phi} = L/I_1$.

As stated above, $L_z = p_\phi$ is true in general:

```
mphi.mtheta.mpsi.L
Collect[%[[3]],{I1,I3},Simplify]
                                                                           2
Out[25]= I3 Cos[theta] (dpsi + dphi Cos[theta]) + dphi I1 Sin[theta]
```

where you'll notice this result is exactly p_ϕ

We have examined here the simplest solutions: those with $\dot{\theta} = 0$ and found (A) a body spinning with $\boldsymbol{\omega}$ and **L** aligned with the 3 axis: a 'statically spinning' object with any orientation, and (B) the motion of a body spinning with $\boldsymbol{\omega}$ not aligned with the 3 axis (and therefore **L** not aligned with either). We are dealing here with free (no-torque) rotations, so of course **L** is always fixed in space. In solution (B) **L** is fixed in the z direction, so the apparent motion of **L** around the 3-axis in the body-fixed frame, must, in the inertial frame become, the 3-axis orbiting around the fixed **L** direction... a spacial wobble. How about the more complex solutions with $\dot{\theta} \neq 0$? These turn out to be cases like (B) but with **L** pointing in a (fixed) direction not aligned with z. The wobbling motion about a non-aligned **L** makes for complex θ dependence.

It's undoubtedly more trouble than it's worth, but as a practice example in the Hamiltonian formulation, let's try to show from the equations that **L** in the inertial frame *is* fixed. The starting point for the Hamiltonian formulation is to express things in terms of canonical momenta (replacing 'velocities' like $\dot{\theta}$). We already have p_ϕ and p_ψ; p_θ is actually the simple one:

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = I_1 \dot{\theta} \tag{15}$$

Solving for the 'velocities' we have:

$$\dot{\phi} = \frac{p_\phi - p_\psi \cos(\theta)}{I_1 \sin^2(\theta)} \tag{16}$$

$$\dot{\psi} = \frac{p_\psi}{I_3} - \frac{p_\phi - p_\psi \cos(\theta)}{I_1 \sin^2(\theta)} \cos(\theta) \tag{17}$$

$$\dot{\theta} = \frac{p_\theta}{I_1} \tag{18}$$

Doing this in *Mathematica* looks odd because of a detail we need to take care of: `pphi` currently means exactly the same thing as Eq. (10); it has no independent status as a variable. We will assign `Pphi` to be that independent variable p_ϕ which has substituted in for Eq. (10).

```
Solve[{Ppsi==ppsi, Pphi==pphi, Ptheta==I1 dtheta},{dphi,dpsi, dtheta}]
subs=First[%]
                                                                  2
                     Ppsi Cot[theta] Csc[theta] - Pphi Csc[theta]
Out[31]=  {dphi -> -(---------------------------------------------),
                                         I1
                                       2
        I1 Ppsi + I3 Ppsi Cot[theta]  - I3 Pphi Cot[theta] Csc[theta]
dpsi -> -------------------------------------------------------------,
                                    I1 I3

          Ptheta
dtheta -> ------}
            I1
```

We now substitute these formulas for 'velocities' into the KE we found earlier:

```
ke /. subs
Simplify[%]
H=Expand[%]
             2         2       2           2
         Ppsi    Ptheta    Ppsi  Cot[theta]
Out[34]= ----- + ------- + ----------------- -
         2 I3     2 I1           2 I1

                                                 2           2
     Pphi Ppsi Cot[theta] Csc[theta]    Pphi  Csc[theta]
>    ------------------------------- + -----------------
                  I1                          2 I1
```

$$H = \frac{p_\theta^2}{2I_1} + \frac{p_\psi^2}{2I_3} + \frac{(p_\psi \cos(\theta) - p_\phi)^2}{I_1 \sin^2(\theta)} \tag{19}$$

We also will need body-frame $\mathbf{L}$ in terms of p_ψ, p_ϕ, p_θ:

```
mphi.mtheta.mpsi.L
% /. subs
{Lx,Ly,Lz}=Simplify[%]
Out[37]= {Cos[phi] (-(Pphi Cot[theta]) + Ppsi Csc[theta]) - Ptheta Sin[phi],

>    Ptheta Cos[phi] + (-(Pphi Cot[theta]) + Ppsi Csc[theta]) Sin[phi], Pphi}
```

$$\mathbf{L} = ((p_\psi - p_\phi \cos\theta)\cos\phi/\sin\theta - p_\theta \sin\theta, (p_\psi - p_\phi \cos\theta)\sin\phi/\sin\theta + p_\theta \cos\theta, p_\phi) \tag{20}$$

Reducing for the moment to a case with just one q and p: if we have a function, $f(q,p)$, of those q,p and we seek its time derivative, we have;

$$\frac{df}{dt} = \frac{\partial f}{\partial q}\,\dot{q} + \frac{\partial f}{\partial p}\,\dot{p} \tag{21}$$

$$= \frac{\partial f}{\partial q}\,\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\,\frac{\partial H}{\partial q} \tag{22}$$

More generally if we have lots of q,p, we simply need to sum all those derivatives:

$$\frac{df}{dt} = \sum_i \left(\frac{\partial f}{\partial q_i}\,\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\,\frac{\partial H}{\partial q_i} \right) \tag{23}$$

$$\equiv [f, H] \tag{24}$$

where the latter generalizes to become the definition for the Poisson bracket $[f,g]$ of any two functions of the q_i, p_i.

So in order to find $\dot{L}_x$, we need to calculate $[L_x, H]$. Doing the sums is not as bad as you might think because H just depends of 4 of the 6 q_i, p_i, and L_x depends on 5 of the 6:

```
D[Lx,theta] D[H,Ptheta] - D[Lx,Ptheta] D[H,theta] + D[Lx,phi] D[H,Pphi]
Simplify[%]
Out[40]= 0
```

So $\dot{L}_x = 0$.

In quantum mechanics the equivalent of Poisson brackets become quite important, so let's calculate a few classically. First the easy ones:

$$[q_i, p_j] = \delta_{ij} \qquad [q_i, q_j] = 0 \qquad [p_i, p_j] = 0 \tag{25}$$

It will require more work to find $[L_i, L_j]$:

```
D[Lx,theta]D[Ly,Ptheta] - D[Lx,Ptheta] D[Ly,theta] +
   D[Lx,phi] D[Ly,Pphi] - D[Lx,Pphi] D[Ly,phi]
Simplify[%]
Out[42]= Pphi

D[Ly,phi] D[Lz,Pphi]
Out[43]= Cos[phi] (-(Pphi Cot[theta]) + Ppsi Csc[theta]) - Ptheta Sin[phi]

-D[Lz,Pphi] D[Lx,phi]
Out[44]= Ptheta Cos[phi] + (-(Pphi Cot[theta]) + Ppsi Csc[theta]) Sin[phi]
```

so $[L_x, L_y] = L_z$, $[L_y, L_z] = L_x$, $[L_z, L_x] = L_y$, or more compactly:

$$[L_i, L_j] = \epsilon_{ijk} L_k \tag{26}$$

How about the components of $\mathbf{L}$ in the body-fixed frame (I'm going to call them K_1, K_2, K_3 so we don't mix them up with the above Ls).

```
L /. subs
{K1,K2,K3}=Simplify[%]
Out[46]= {Cos[psi] (-Pphi + Ppsi Cos[theta]) Csc[theta] + Ptheta Sin[psi],

>    Ptheta Cos[psi] + (-(Ppsi Cot[theta]) + Pphi Csc[theta]) Sin[psi], Ppsi}
```

$$\mathbf{K} = (-(p_\phi - p_\psi \cos\theta)\cos\psi/\sin\theta + p_\theta \sin\psi, (p_\phi - p_\psi \cos\theta)\sin\psi/\sin\theta + p_\theta \cos\psi, p_\psi) \tag{27}$$

```
D[K1,theta] D[K2,Ptheta] - D[K1,Ptheta] D[K2,theta] +
    D[K1,psi] D[K2,Ppsi] - D[K1,Ppsi] D[K2,psi]
Simplify[%]
Out[48]= -Ppsi

D[K2,psi]D[K3,Ppsi]
Out[49]= Cos[psi] (-(Ppsi Cot[theta]) + Pphi Csc[theta]) - Ptheta Sin[psi]

-D[K3,Ppsi] D[K1,psi]
Out[50]= -(Ptheta Cos[psi]) + (-Pphi + Ppsi Cos[theta]) Csc[theta] Sin[psi]
```

so $[K_1, K_2] = -K_3$, $[K_2, K_3] = -K_1$, $[K_3, K_1] = -K_2$, or more compactly:

$$[K_i, K_j] = -\epsilon_{ijk} K_k \tag{28}$$

Finally we should expect that H should have a simple, understandable form in terms of K_i:

```
K1^2+K2^2
Expand[%]
Simplify[%]
                        2        2             2
Out[54]= Ptheta  + Ppsi  Cot[theta]  - 2 Pphi Ppsi Cot[theta] Csc[theta] +

             2             2
>    Pphi  Csc[theta]
```

Notice that these exact same terms occur in H so the result is:

$$H = \frac{K_1^2 + K_2^2}{2I_1} + \frac{K_3^2}{2I_3} \tag{29}$$

Since $L_z = p_\phi$ and there are no raw ϕ in $\mathbf{K}$, we conclude $[L_z, \mathbf{K}] = 0$. Similarly since $K_3 = p_\psi$ and there are no raw ψ in $\mathbf{L}$, we conclude $[K_3, \mathbf{L}] = 0$. How about the others e.g., $[L_x, \mathbf{K}]$?

The product rule can be used to show:

$$[A, B\, C] = [A, B]C + B[A, C] \tag{30}$$

which allows easy calculation (i.e., *Mathematica* not needed) of $\dot{K}_i = [K_i, H]$:

$$\begin{aligned}
\dot{K}_3 &= [K_3, H] = \frac{1}{2I_1}\,(2[K_3, K_1]K_1 + 2[K_3, K_2]K_2) = 0 && (31)\\
\dot{K}_1 &= \frac{1}{2I_1}\,2[K_1, K_2]K_2 + \frac{1}{2I_3}\,2[K_1, K_3]K_3 = \left(\frac{1}{I_3} - \frac{1}{I_1}\right) K_3K_2 = \Omega_b K_2 && (32)\\
\dot{K}_2 &= \frac{1}{2I_1}\,2[K_2, K_1]K_1 + \frac{1}{2I_3}\,2[K_2, K_3]K_3 = -\left(\frac{1}{I_3} - \frac{1}{I_1}\right) K_3K_1 = -\Omega_b K_1 && (33)
\end{aligned}$$

From the first we can conclude that K_3 is constant (and further since H is constant, $K_1^2 + K_2^2$ must also be constant). The following two show SHO at the angular frequency Ω_b as in Eqs.(13–14). So the $\mathbf{K}$ vector traces a cone at angular frequency Ω_b and with constant 3 component.