$$\begin{aligned} 2T &= I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \\ L^2 &= I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 \\ I_i\dot{\omega}_i &= (I_j - I_k)\ \omega_j\omega_k \qquad (ijk) \text{ cyclic } (123) \end{aligned}$$

$$\text{For } \omega_3 \gg \omega_1, \omega_2 \Rightarrow \omega_3 \sim \text{constant and } \ddot{\omega}_1 \approx \underbrace{\frac{(I_2 - I_3)(I_3 - I_1)}{I_1 I_2}\,\omega_3^2}_{\text{If } < 0 \text{ then stable; } \equiv -\Omega_b^2}\ \omega_1$$

$$\Omega_b = \sqrt{\frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}}\ \omega_3 \xrightarrow{I_1 = I_2} \frac{|I_3 - I_1|}{I_1}\ \omega_3 \xrightarrow{I_3 = 2I_1} \omega_3$$

$$\left.\begin{array}{l} \text{If } \dot{\omega}_2 \propto +\omega_1 \text{ then countercockwise} \\ \text{If } \dot{\omega}_2 \propto -\omega_1 \text{ then cockwise} \end{array}\right\} \Omega_b = \frac{I_3 - I_1}{I_1}\ \omega_3$$

Ω_b is frequency of $\mathbf{L}$ & $\boldsymbol{\omega}$ circling about 3-axis in body frame
$\dot{\phi}$ is wobble frequency of 3-axis (and $\boldsymbol{\omega}$) circling about $\mathbf{L}$ in inertial frame.

Non-trivial Euler angle solution $\theta =$constant: $\cos\theta = p_\psi/p_\phi = L_3/L_z = L_3/L$

$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta} \to \frac{p_\phi(1-\cos^2\theta)}{I_1\sin^2\theta} = \frac{L_z}{I_1} = \frac{p_\psi}{I_1\cos\theta} = \frac{I_3\omega_3}{I_1\cos\theta} \xrightarrow{\theta \ll 1; I_3 = 2I_1} 2\omega_3$$

$$L = \frac{1}{2}\ I_1\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}\ I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 - mgR\cos\theta$$

$$\begin{aligned} p_\psi &= \frac{\partial T}{\partial\dot{\psi}} = I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right) = L_3 \equiv I_1 a \\ p_\phi &= \frac{\partial T}{\partial\dot{\phi}} = I_3\cos\theta\left(\dot{\phi}\cos\theta + \dot{\psi}\right) + I_1\ \dot{\phi}\sin^2\theta = p_\psi\cos\theta + I_1\ \dot{\phi}\ \sin^2\theta = L_z \equiv I_1 b \\ \dot{\phi} &= \frac{b - a\cos\theta}{\sin^2\theta} \end{aligned}$$

$$mgR/I_1 \equiv c^2$$

$$E' = \frac{E}{I_1} = \frac{1}{2}\ \dot{\theta}^2 + \frac{(b - a\cos\theta)^2}{2\sin^2\theta} + c^2\cos\theta + \frac{I_1 a^2}{2I_3} = \frac{1}{2}\ \dot{\theta}^2 + V(\theta) + \text{constant} \tag{1}$$

$$V(u = \cos\theta) = \frac{(b - au)^2}{2(1-u^2)} + c^2 u \tag{2}$$

$$\begin{aligned} 0 = \frac{dV}{du} &= -a\ \frac{(b - au)}{(1-u^2)} + u\ \frac{(b-au)^2}{(1-u^2)^2} + c^2 \\ &= -a\dot{\phi} + u\dot{\phi}^2 + c^2 \end{aligned}$$

$$\theta'' - \cot\theta\csc^2\theta\left(a^2 + b^2 - ab(\cos\theta + \sec\theta)\right) - c^2\sin\theta = 0 \tag{3}$$

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} \tag{4}$$