Euler Eqs: $\vec{\Gamma} = \underbrace{\frac{d\vec{L}}{dt}}_{\text{inertial}} = \underbrace{\frac{d\vec{L}}{dt} + \overrightarrow{\omega \times L}}_{\text{rotating}}$

if torque $\vec{\Gamma} = 0$ & use principal axes in body frame:

$$\vec{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$$

$$\omega_2\omega_3(I_2 - I_3)$$

$$\overrightarrow{\omega \times L} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3 \end{vmatrix} = \left( \omega_2\omega_3(I_3 - I_2),\ \omega_1\omega_3(I_1 - I_3),\ \hat{}\ \right)$$

so

$$I_1\dot{\omega}_1 = \omega_2\omega_3(I_2 - I_3)$$
$$I_2\dot{\omega}_2 = \omega_1\omega_3(I_3 - I_1)$$
$$I_3\dot{\omega}_3 = \omega_1\omega_2(I_1 - I_3)$$

Note: coordinate on left is NOT on right. In the I differences the order is cyclic: 1 = 2 3, 2 = 3 1, 3 = 1 2

Consider an initial state where $\omega_3 \gg \omega_1$ & $\omega_2$. If we "neglect" $\omega_1\omega_2$ (ie assume close enough to zero)

$\Rightarrow$ $I_3\dot{\omega}_3 = 0 \longrightarrow \omega_3 = \text{constant}$

Now differentiate the first eq & sub in the second.

$$I_1\ddot{\omega}_1 = (I_2 - I_3)\underset{\substack{\uparrow \\ \text{assumed} \\ \text{constant}}}{\omega_3}\dot{\omega}_2 = (I_2 - I_3)\omega_3\frac{\omega_1\omega_3}{I_2}(I_3 - I_1)$$

$$\ddot{\omega}_1 = \underset{\substack{\uparrow \\ \text{yes}}}{-}\underbrace{\frac{(I_3 - I_2)(I_3 - I_1)\omega_3^2}{I_1 I_2}}\omega_1$$

call it $\Omega^2$

If this is positive we have SHO

Exactly the same process yields exactly the same eq except for $\omega_2 \longrightarrow \omega_1$ & $\omega_2$ have SHO at same freq

IF $I_3 > I_1, I_2$ & $\omega_1 \sim \cos\Omega t$ then $\omega_2 \sim \sin\Omega t$

IF $I_3 < I_1, I_2$ & $\omega_1 \sim \cos\Omega t$ then $\omega_2 \sim -\sin\Omega t$

IF $I_3$ between $I_1$ & $I_2$ → the constant is $-$

If that constant is $\ominus$ $\Rightarrow$ $\ddot{\omega}_1 = +\lambda^2 \omega_1$

$\Rightarrow$ $\omega_1 \sim e^{+\lambda t}$ & $e^{-\lambda t}$

As a result $\omega_1$ & $\omega_2$ will not remain small $\rightarrow$ rotation about 3 axis will not be stable.

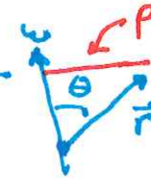
"Tennis Racket" Thm – stable rotations about largest & smallest $I$ – rotations about axis with intermediate $I$ $\Rightarrow$ tumble not rotation.

Calculation of KE: $T = \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega} = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$

using principal axes

$$\frac{1}{2}\sum m_\alpha \vec{\omega}\cdot\left\{ r_\alpha^2 \overleftrightarrow{1} - \vec{r}_\alpha \vec{r}_\alpha \right\}\cdot\vec{\omega}$$

$$= \frac{1}{2}\sum m_\alpha \left\{ r_\alpha^2 - r_\alpha^2 \cos^2\theta_\alpha \right\} \omega^2$$




From dot product

$$= \frac{1}{2}\sum m_\alpha r_\alpha^2 \left\{ 1 - \cos^2\theta_\alpha \right\} \omega^2$$

$$= \frac{1}{2}\sum m_\alpha r_\alpha^2 \sin^2\theta_\alpha \,\omega^2 = \frac{1}{2}\sum m_\alpha \rho_\alpha^2\, \omega^2$$

Remark: can also write $T = \frac{1}{2}\left(\frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3}\right)$

so if rotating (stably) about small $I_1$ axis thats a relatively high energy state. Given time and friction expect rotation about large $I$ axis.

Eg satellites


Fast on axis spin

Slow spin ⊥ to axis.

ie $\vec{L}$ will be conserved

$E$ gets reduced due to "friction"

Symmetric objects: $I_1 = I_2 \Rightarrow \dot{\omega}_3 = 0$ ($\omega_3 = $ const)

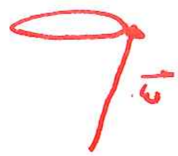
previous work $\rightarrow$ $\ddot{\omega}_1 = - \underbrace{\frac{(I_3 - I_1)^2 \omega_3^2}{I_1^2}}_{} \omega_1$

$\Omega = \frac{|I_3 - I_1| \omega_3}{I_1}$

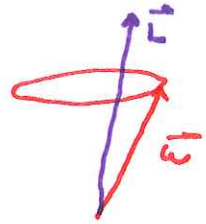
If $I_3 > I_1, I_2 \Rightarrow$

$I_3 < I_1, I_2 \Rightarrow$

So in body frame:

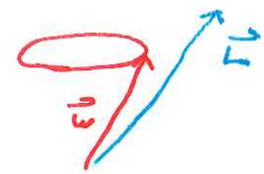



$\vec{\omega}$ goes in cone ($\omega_3$ fixed, $\omega_1$ & $\omega_2$ circle)




IF $I_3 > I_1, I_2$

IF $I_3 < I_1 I_2$

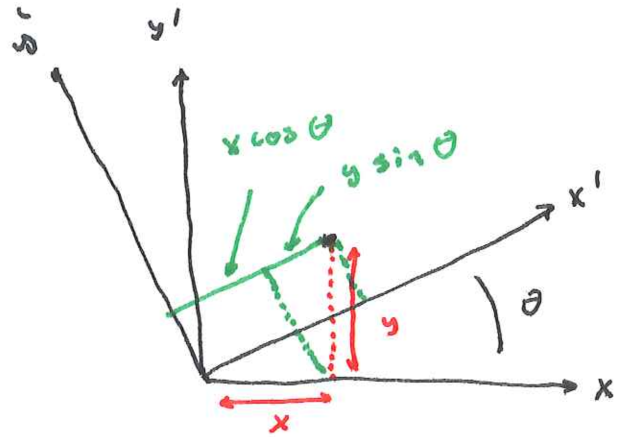



Note: $\vec{L}$, $\vec{\omega}$ & $\hat{e}_3$ are co planar; $\vec{L}$ & $\vec{\omega}$ rotate together

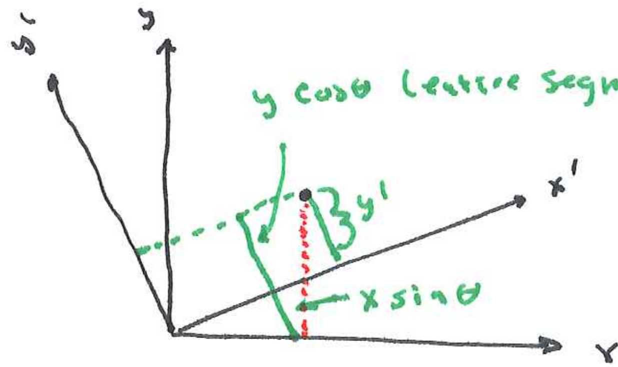
IF try to picture in space (inertial) frame – $\vec{L}$ is fixed & body $\hat{e}_3$ axis rotates about it ... "wobble". You might guess that the space wobble frequency is same as body wobble freq ... not so ... we'll show the space wobble freq $= L/I_1$

Start Euler Angles

2d rotation matrices —




$x' = \cos\theta\, x + \sin\theta\, y$




$y' = y\cos\theta - x\sin\theta$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Eg the coordinates of the point $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in primed frame

are $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta + \sin\theta \\ -\sin\theta + \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Note: For this rotation matrix: $M^T = M^{-1} = M(-\theta)$

$\det(M) = 1$ → These are the general properties of rotation matrices in particular in 3d space

... called SO(3)

orthogonal

special, ie $\det M = 1$