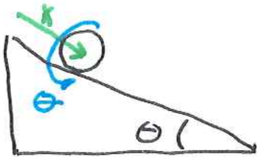
More Lagrange examples —

rolling down hill



Note: arc length along "ball" = $R\theta$ = distance x

$R\omega = \dot{x}$

$$KE = \underbrace{\tfrac{1}{2} m\dot{x}^2}_{\text{"of CM"}} + \overbrace{\tfrac{1}{2} I \omega^2}^{\text{about CM}}$$

I about CM $\equiv \gamma M R^2$

object	γ
ball	2/5
disk	1/2
hoop	1
shell	2/3

$PE = -mgx\sin\theta$

$$L = KE - PE = \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2}\gamma M R^2 \omega^2 + mgx\sin\theta$$

$\omega = \frac{\dot{x}}{R}$

$$= m\left\{ \tfrac{1}{2}(1+\gamma)\dot{x}^2 + g\sin\theta\, x \right\}$$

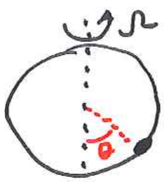
Note: overall factor in L does not affect min/max condition

$$\frac{\partial L}{\partial \dot{x}} = m(1+\gamma)\dot{x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m(1+\gamma)\ddot{x} = \frac{\partial L}{\partial x} = mg\sin\theta$$

so $\ddot{x} = \dfrac{g\sin\theta}{(1+\gamma)}$

Bead on a rotating circle



$\downarrow g$

Note: in spherical coordinates:

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi}$$

radial velocity — North/South velocity — East/West velocity

In this problem $\dot{\phi}$ = constant = Ω

and r = constant = R

$$\vec{v} = R\dot{\theta}\hat{\theta} + R\sin\theta\,\Omega\hat{\phi}$$

$$v^2 = (R\dot{\theta})^2 + (R\sin\theta\,\Omega)^2$$

$PE = -mgR\cos\theta$

$KE = \tfrac{1}{2} m R^2 \left[\dot{\theta}^2 + \sin^2\theta\,\Omega^2 \right]$

$$L = KE - PE = \underbrace{mR^2}_{\text{drop}} \left\{ \tfrac{1}{2}\left[\dot{\theta}^2 + \sin^2\theta\,\Omega^2\right] + \underbrace{\frac{g}{R}}_{\text{define as } \omega^2} \cos\theta \right\}$$

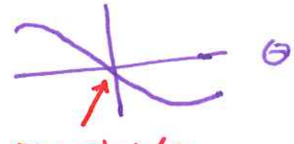
$$\frac{\partial L}{\partial \dot{\theta}} = \dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \ddot{\theta} = \frac{\partial L}{\partial \theta} = \sin\theta\cos\theta\,\Omega^2 - \omega^2\sin\theta$$

$$\ddot{\theta} = -\omega^2\sin\theta\left\{1 - \frac{\Omega^2}{\omega^2}\cos\theta\right\}$$

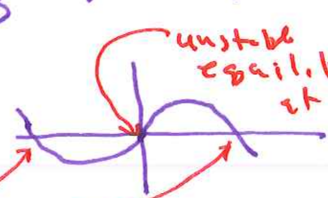
$-\omega^2 \sin\theta \left\{ 1 - \frac{\Omega^2}{\omega^2}\cos\theta \right\} = f(\theta)$

if $\Omega < \omega$ then $\{\,\} > 0$ & above quantity looks like



one stable equilibrium @ $\theta = 0$

if $\Omega > \omega$ then $\{\,\}$ switches sign at $\cos\theta = \frac{\omega^2}{\Omega^2}$



unstable equilibrium at $\theta = 0$

stable equilib. @ $\cos\theta = \frac{\omega^2}{\Omega^2}$

small amplitude oscillation frequency: Taylor expand above @ an equilibrium position –

$f'(\theta=0) = -\omega^2\left\{1 - \frac{\Omega^2}{\omega^2}\right\} = -\left\{\omega^2 - \Omega^2\right\}$

$\ddot{\theta} = f(\theta) \approx f'(\theta=0)\,\theta \Rightarrow \tilde{\omega} = \sqrt{\omega^2 - \Omega^2}$

$f'\left(\cos\theta_0 = \frac{\omega^2}{\Omega^2}\right) = -\omega^2 \sin\theta_0 \left(+\frac{\Omega^2}{\omega^2}\sin\theta_0\right)$

$= -\Omega^2 \sin^2\theta_0 = -\Omega^2\left(1 - \cos^2\theta_0\right) = -\Omega^2\left(1 - \left(\frac{\omega^2}{\Omega^2}\right)^2\right)$

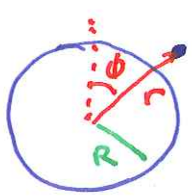
$= -\left(\Omega^2 - \frac{\omega^4}{\Omega^2}\right) \Rightarrow \tilde{\omega} = \sqrt{\Omega^2 - \frac{\omega^4}{\Omega^2}}$

in our textbook notation

$\Omega \to \omega$

$\omega^4 \to \left(\frac{g}{R}\right)^2$

Constraints – slide down sphere



constraining function : $f(r,\phi) = r = R$

$KE = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right)$ $\quad PE = mgr\cos\phi$

$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - mgr\cos\phi$

r: $\quad m\ddot{r} = mr\dot{\phi}^2 - mg\cos\phi\ \cancel{mg\cos\phi} + \lambda\cdot 1 \quad \frac{\partial f}{\partial r}$

ϕ: $\quad \frac{d}{dt}\left(mr^2\dot{\phi}\right) = mgr\sin\phi + \lambda\cdot 0 \quad \frac{\partial f}{\partial \phi}$

Note: r is not considered constant yet – when we solve we may apply the constraint

Solve: $m\ddot{r} = m r\dot{\phi}^2 - mg\cos\phi + \lambda \Rightarrow 0 = mR\,2\omega^2(1-\cos\phi) - mg\cos\phi + \lambda$

$$\left.\begin{aligned} &\frac{d}{dt}(mr^2\dot{\phi}) = gmr\sin\phi \\ &r = R \end{aligned}\right\} \quad \ddot{\phi} = \frac{g}{R}\sin\phi$$

$\frac{g}{R}$ → define as ω^2

integrating factor: $\dot{\phi}$

$$\dot{\phi}\ddot{\phi} = \left(\tfrac{1}{2}\dot{\phi}^2\right)^{\cdot} = \omega^2\left(-\cos\phi\right)^{\cdot} = \omega^2\sin\phi\,\dot{\phi}$$

$$\frac{d}{dt}\Big(\underbrace{\tfrac{1}{2}\dot{\phi}^2 + \omega^2\cos\phi}_{\text{from initial conditions this constant must be } \omega^2}\Big) = 0$$

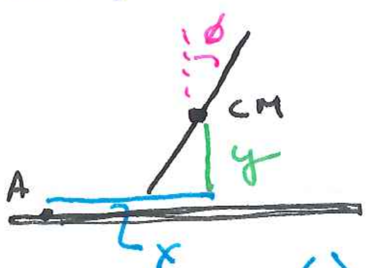
$$\tfrac{1}{2}\dot{\phi}^2 = \omega^2(1-\cos\phi)$$

$$\lambda = mg\cos\phi - mR\,2\,\frac{g}{R}(1-\cos\phi)$$

$$= mg(3\cos\phi - 2)$$

$\hookrightarrow = 0$ at $\cos\phi = 2/3$

Falling Ladder problem → 2 constraints & 2 Lagrange Mults



$$KE = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) + \tfrac{1}{2}I_{CM}\dot{\phi}^2, \quad I_{CM} = \tfrac{1}{3}mR^2$$

$$PE = mgy$$

Lagrange Multipliers:
- λ constraint ①: $f(x,y,\phi) = x - R\sin\phi = 0$
- μ constraint ②: $g(x,y,\phi) = y - R\cos\phi = 0$

(x) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} = \frac{\partial L}{\partial x} + \lambda\frac{\partial f}{\partial x} + \mu\frac{\partial g}{\partial x} = \lambda$$

(y) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = m\ddot{y} = \frac{\partial L}{\partial y} + \lambda\frac{\partial f}{\partial y} + \mu\frac{\partial g}{\partial y} = -mg + 0 + \mu$$

(ϕ) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = I_{CM}\ddot{\phi} = \frac{\partial L}{\partial \phi} + \lambda\frac{\partial f}{\partial \phi} + \mu\frac{\partial g}{\partial \phi} = 0 + -\lambda R\cos\phi + \mu R\sin\phi$$

Solve: $m\ddot{x} = \lambda \longrightarrow mR\,(\ddot{\sin\phi}) = \lambda$

$m\ddot{y} = -mg + \mu \rightarrow mR\,(\ddot{\cos\phi}) = -mg + \mu$

$I_{cm}\,\ddot{\phi} = -\lambda R\cos\phi + \mu R\sin\phi \rightarrow \frac{1}{3}MR^2\ddot{\phi} = -\lambda R\cos\phi + \mu R\sin\phi$

$x = R\sin\phi$

$y = R\cos\phi$

Reduce algebra! divide first two eqns by mR and define $\tilde{\lambda} = \frac{\lambda}{mR}$ $\tilde{\mu} = \frac{\mu}{mR}$ $\omega^2 = \frac{g}{R}$

Divide third eqn by MR^2

Define $C \equiv \cos\phi$, $S \equiv \sin\phi$ $\rightarrow$ $\dot{c} = -s\dot{\phi}$, $\dot{s} = c\dot{\phi}$ $\longrightarrow$ $\ddot{c} = -c\dot{\phi}^2 - s\ddot{\phi}$, $\ddot{s} = -s\dot{\phi}^2 + c\ddot{\phi}$

$$\ddot{s} = -s\dot{\phi}^2 + c\ddot{\phi} = \tilde{\lambda}$$

$$\ddot{c} = -c\dot{\phi}^2 - s\ddot{\phi} = -\omega^2 + \tilde{\mu}$$

$$\frac{1}{3}\ddot{\phi} = -\tilde{\lambda}c + \tilde{\mu}s = -(-s\dot{\phi}^2 + c\ddot{\phi})c + (\omega^2 - c\dot{\phi}^2 - s\ddot{\phi})s$$

$-(c^2+s^2)\ddot{\phi}$

cancel

$$\frac{4}{3}\ddot{\phi} = \omega^2 s \xrightarrow[\text{fact } \dot{\phi}]{\text{integrating}} \frac{2}{3}\frac{d}{dt}\dot{\phi}^2 = -\omega^2\frac{d}{dt}c$$

$$\frac{d}{dt}\Big(\underbrace{\frac{2}{3}\dot{\phi}^2 + \omega^2 c}_{\text{constant} = \omega^2}\Big) = 0$$

$$\frac{2}{3}\dot{\phi}^2 = \omega^2(1-c)$$

Now: $\tilde{\mu} = \omega^2 - c\dot{\phi}^2 - s\ddot{\phi}$

$$= \omega^2 - c\,\frac{3}{2}\omega^2(1-c) - s\,\frac{3}{4}\omega^2 s \qquad s^2 = 1-c^2$$

$$= \omega^2\left\{1 - \frac{3}{2}c + \frac{3}{2}c^2 - \frac{3}{4} + c^2\frac{3}{4}\right\}$$

$$= \omega^2\left\{\frac{1}{4} - \frac{3}{2}c + \frac{9}{4}c^2\right\}$$

$$= \omega^2\left(\frac{1}{2} - \frac{3}{2}c\right)^2$$

$\longrightarrow$ normal force μ goes to zero at $c = \frac{1}{3}$

$$\tilde{\lambda} = -s\,\dot{\phi}^2 + c\,\ddot{\phi}$$

$$= -s\,\tfrac{3}{2}\,\omega^2(1-c) + c\,\tfrac{3}{4}\,\omega^2 s$$

$$= -s\,\tfrac{3}{2}\,\omega^2 \left\{ (1-c) - \tfrac{1}{2}c \right\}$$

$$= -s\,\tfrac{3}{2}\,\omega^2 \left\{ 1 - \tfrac{3}{2}c \right\}$$

changes sign at $c = 2/3$

$$\frac{F}{N} = \mu = \left|\frac{\lambda}{\mu}\right| = \frac{s\,\tfrac{3}{2}\left|\tfrac{3}{2}c - 1\right|}{\left(\tfrac{1}{2} - \tfrac{3}{2}c\right)^2} = \frac{6s\left|\tfrac{3}{2}c - 1\right|}{(1-3c)^2}$$

coef of static friction here