Our textbook, and most physics textbooks[1], discusses only "holonomic constraints" which essentially are constraints where the physical coordinates can be expressed as functions of the generalized coordinates. For example if constrained to move in a circle of radius R on a plane:

$$\begin{aligned} x &= R\cos\theta && (1)\\ y &= R\sin\theta && (2)\\ z &= 0 && (3) \end{aligned}$$

so we can eliminate (x, y, z) in favor of the generalized coordinate θ. However a disk rolling on a plane can 'drive' all over the surface with no fixed relation between location of the contact point P and the current orientation of the disk. These nonholonomic constraints can fit into Lagrange's formulation, but undergraduate physics sees better education targets elsewhere. So in what follows we'll avoid overwhelmingly complete new techniques in favor of using available techniques and approximations to investigate the motion of rolling objects— but this 'simple' (no new techniques) approach can turn out to be a real mess.

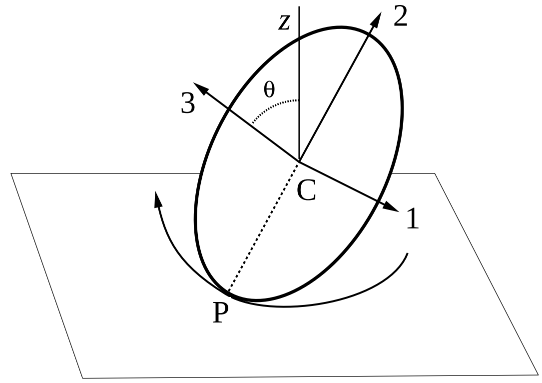

Figure 1: Coordinate frame for rolling disk. Axis 3 is in the direction of the disk's axle; 1 is always parallel to the plane and in the trailing direction if $\dot\psi > 0$; 2 points away from the contact point P.

We begin with a rolling disk (radius R) whose orientation is described by Euler angles (ϕ, θ, ψ). Because a disk is rotationally symmetric, we can avoid a transformation all the way to the body-fixed frame and instead use a frame in which the disk is spinning (at $\dot\psi$); this is the coordinate frame just before the step to the final 'body-fixed' frame. The direction 3 is the 'axle' of the disk; 1 is parallel to the plane and, in this simple case, 2 points directly away from the contact point P. While I will refer to the object as a 'disk' I mean that to include other symmetric objects, e.g., a hoop. Note that since $2I_1 = I_3$ for plane symmetric figures, different I can be parametrized by the fraction in front of mR^2: $k = \frac{1}{2}$ for a hoop; $k = \frac{1}{4}$ for a disk. Again, while the disk is spinning in this frame, the moment of inertial $\mathbf{I}$ is constant in this frame:

$$\mathbf{I} = \begin{pmatrix} I_1 & 1 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \qquad \mathbf{I}_{\text{disk}} = \frac{1}{4}\, mR^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \mathbf{I}_{\text{hoop}} = \frac{1}{2}\, mR^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad (4)$$

To find the equations of motion, note that $d\mathbf{L}/dt$ about the CM must equal the torque, $\mathbf{\Gamma}$, calculated about the CM. We define the vector $\overrightarrow{CP} = \mathbf{r}_P = -R\mathbf{e}_2$ and let $\mathbf{F}$ be the contact force at P so the torque is $\mathbf{r}_P \times \mathbf{F}$. The calculation of $d\mathbf{L}/dt$ in the 123 frame must account for the rotation, $\mathbf{\Omega}$, of that frame:

$$\left.\frac{d\mathbf{L}}{dt}\right|_{123} + \mathbf{\Omega} \times \mathbf{L} = \mathbf{r}_P \times \mathbf{F} \quad (5)$$

[1]Symon: "Nonholonomic constraints occur in some problems in which bodies roll without slipping, but they are not of great importance in physics. We shall therefore restrict attention to holonomic systems."

We will calculate below $\mathbf{L}$ and $\mathbf{\Omega} = \boldsymbol{\omega} - \dot{\psi}\mathbf{e}_3$ using Euler angles, but first we need to consider the motion of the CM:

$$m\dot{\mathbf{v}}_C = m\left(\left.\frac{d\mathbf{v}_C}{dt}\right|_{123} + \mathbf{\Omega}\times\mathbf{v}_C\right) = \mathbf{F} + m\mathbf{g} \tag{6}$$

By the no-slip rolling condition: $\mathbf{v}_C = -\boldsymbol{\omega}\times\mathbf{r}_P$. Since $\mathbf{r}_P$ is a constant in 123, we have:

$$m\left(-\dot{\boldsymbol{\omega}}\times\mathbf{r}_P + \mathbf{\Omega}\times\mathbf{v}_C\right) = \mathbf{F} + m\mathbf{g} \tag{7}$$

Eliminating $\mathbf{F}$ gives:

$$\mathbf{I}\cdot\dot{\boldsymbol{\omega}} + \mathbf{\Omega}\times\mathbf{L} = m\mathbf{r}_P\times\left(-\dot{\boldsymbol{\omega}}\times\mathbf{r}_P + \mathbf{\Omega}\times\mathbf{v}_C - \mathbf{g}\right) \tag{8}$$

Renaming $\mathbf{I}\leftarrow\mathbf{I}/m$ gives:

$$\begin{aligned}
\mathbf{I}\cdot\dot{\boldsymbol{\omega}} + \mathbf{\Omega}\times(\mathbf{I}\cdot\boldsymbol{\omega}) &= \mathbf{r}_P\times\left(-\dot{\boldsymbol{\omega}}\times\mathbf{r}_P + \mathbf{\Omega}\times(\mathbf{r}_P\times\boldsymbol{\omega}) - \mathbf{g}\right) && (9)\\
\mathbf{I}\cdot\dot{\boldsymbol{\omega}} + \mathbf{r}_P\times(\dot{\boldsymbol{\omega}}\times\mathbf{r}_P) &= -\mathbf{\Omega}\times(\mathbf{I}\cdot\boldsymbol{\omega}) + \mathbf{r}_P\times\left(\mathbf{\Omega}\times(\mathbf{r}_P\times\boldsymbol{\omega}) - \mathbf{g}\right) && (10)\\
\mathbf{I}\cdot\dot{\boldsymbol{\omega}} + R^2\dot{\boldsymbol{\omega}} - R^2\dot{\omega}_2\mathbf{e}_2 &= -\mathbf{\Omega}\times(\mathbf{I}\cdot\boldsymbol{\omega}) + \mathbf{r}_P\times\left(R\Omega_2\boldsymbol{\omega} - \mathbf{g}\right) && (11)
\end{aligned}$$

Renaming $\mathbf{I}\leftarrow\mathbf{I}/R^2$ and $g\leftarrow g/R$ gives:

$$\mathbf{I}\cdot\dot{\boldsymbol{\omega}} + \dot{\boldsymbol{\omega}} - \dot{\omega}_2\mathbf{e}_2 = -\mathbf{\Omega}\times(\mathbf{I}\cdot\boldsymbol{\omega}) - \mathbf{e}_2\times\Omega_2\boldsymbol{\omega} - \mathbf{e}_1\ \cos\theta\ g \tag{12}$$

The coordinate system used is related to a 313 Euler angle set. In our notation:

$$\begin{aligned}
\mathbf{\Omega} &= \begin{pmatrix}\dot{\theta}\\0\\0\end{pmatrix} + \mathcal{M}_\theta^{-1}\begin{pmatrix}0\\0\\\dot{\phi}\end{pmatrix} && (13)\\
&= \left(\dot{\theta},\ \dot{\phi}\sin\theta,\ \dot{\phi}\cos\theta\right) && (14)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\omega} &= \begin{pmatrix}0\\0\\\dot{\psi}\end{pmatrix} + \begin{pmatrix}\dot{\theta}\\0\\0\end{pmatrix} + \mathcal{M}_\theta^{-1}\begin{pmatrix}0\\0\\\dot{\phi}\end{pmatrix} && (15)\\
&= \left(\dot{\theta},\ \dot{\phi}\sin(\theta),\ \dot{\phi}\cos(\theta) + \dot{\psi}\right) && (16)
\end{aligned}$$

Eq. (12) now reads

$$\begin{pmatrix}(I_1+1)\,\dot{\omega}_1\\ I_1\,\dot{\omega}_2\\ (I_3+1)\,\dot{\omega}_3\end{pmatrix} = \begin{pmatrix}-(\dot{\psi}(1+I_3) + \dot{\phi}(1+I_3-I_1)\cos\theta)\dot{\phi}\sin\theta - g\cos\theta\\ \dot{\theta}(\dot{\psi}I_3 + \dot{\phi}(I_3-I_1)\cos\theta)\\ \dot{\phi}\dot{\theta}\sin\theta\end{pmatrix} \tag{17}$$

or using the shorthand $I_1 = k; I_3 = 2k$ we have

$$\begin{pmatrix}(k+1)\,\dot{\omega}_1\\ k\,\dot{\omega}_2\\ (2k+1)\,\dot{\omega}_3\end{pmatrix} = \begin{pmatrix}-(\dot{\psi}(2k+1) + \dot{\phi}(k+1)\cos\theta)\dot{\phi}\sin\theta - g\cos\theta\\ k\dot{\theta}(2\dot{\psi} + \dot{\phi}\cos\theta)\\ \dot{\phi}\dot{\theta}\sin\theta\end{pmatrix} \tag{18}$$

When you produce a mess like this it is essential that you do some checking: finding physical situations that you independently know are solutions and checking that they also solve the differential equations.

Totally static sitting on the ground ($\theta = 0$) isn't an option since we have forced the contact force to be on an edge.

'On a knife edge' (unstable?) $\theta = \pi/2$, and either straight-line rolling ($\dot\psi \neq 0, \dot\phi = 0$) or spinning on a diameter ($\dot\psi = 0, \dot\phi \neq 0$)... yes the rhs of Eq. (18) is zero.

The first component of Eq. (18) is substantially tested by considering a disk where the contact point P traces a circle of radius ρ. The torques from the contact precess the angular momentum at exactly the same angular rate as CM moves: $\dot\phi$. The angular momentum about the CM comes in two pieces: in the 3 direction (mostly due to $\dot\psi$) and in the 2 direction due to the changing orientation of the disk ($\dot\phi$). This vector has constant magnitude but its horizontal component $L_\perp$ is changing direction. That change in direction must be due to the torques (calculated about the CM) produced by the contact forces. Since the CM keeps constant height (as θ is constant), the vertical component of the contact force: $N = mg$. The horizontal contact force must provide the centripetal force to move the CM in a circle of radius $\rho - R\sin\alpha$. Noting Eq. (16) see that:

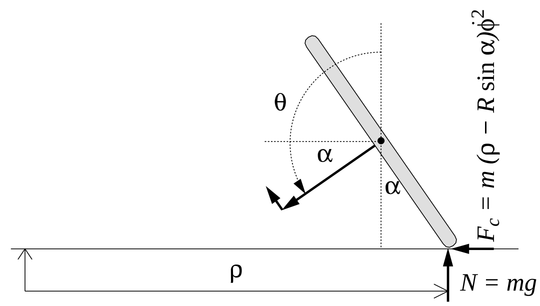

Figure 2: A disk moves in a circle (radius ρ) at constant θ, where the forces/torques are designed to rotated the CM and precess **L** at exactly the same constant rate: $\dot\phi$

$$L_\perp = I_3(\dot\psi + \dot\phi\cos\theta)\cos\alpha + I_1\dot\phi\sin\theta\sin\alpha \tag{19}$$

$d\mathbf{L}/dt$ is then out of the page with magnitude $L_\perp\dot\phi$. The net torque (out of the page) is:

$$\Gamma = mgR\sin\alpha - m(\rho - R\sin\alpha)\dot\phi^2 R\cos\alpha \tag{20}$$

Note that $\theta = \alpha + \pi/2$ so $\sin\alpha = -\cos\theta$ and $\cos\alpha = \sin\theta$ and that given $\theta > \pi/2$ both of these quantities are positive. Finally we shall use in one place: $\dot\psi = \rho\dot\phi/R$ which is required by the equality of the path lengths: $\rho\phi = R\psi$. (In the second line we make the replacements $I \leftarrow I/mR^2$, $g \leftarrow g/R$, and $\dot\psi = \rho\dot\phi/R$.)

$$\begin{aligned}
-mgR\cos\theta - m(\rho + R\cos\theta)\dot\phi^2 R\sin\theta &= \Big(I_3(\dot\psi + \dot\phi\cos\theta)\sin\theta - I_1\dot\phi\sin\theta\cos\theta\Big)\,\dot\phi \\
-g\cos\theta - (\dot\psi + \dot\phi\cos\theta)\dot\phi\sin\theta &= \Big(I_3(\dot\psi + \dot\phi\cos\theta) - I_1\dot\phi\cos\theta\Big)\,\dot\phi\sin\theta \\
-g\cos\theta - (\dot\psi(1 + I_3) + (1 + I_3 - I_1)\dot\phi\cos\theta)\dot\phi\sin\theta &= 0
\end{aligned} \tag{21}$$

Compare to component 1 of Eq. (17).

Much the same diagram (but with $\dot\phi = 0, \dot\psi = 0$) can describe a disk tipping over. Calculating gravity's torque about the contact point, and using the parallel axis theorem for that rotation axis:

$$(I_1 + mR^2)\,\ddot{\alpha} = mgR\,\sin\alpha \tag{22}$$
$$(I_1 + mR^2)\,\dot{\omega}_1 = -mgR\,\cos\theta \tag{23}$$
$$(I_1 + 1)\,\dot{\omega}_1 = -g\cos\theta \tag{24}$$

where in the last line we have scaled out mR^2. Compare to component 1 of Eq. (17).

Some coin motions 'on a knife edge' are familiar because they can be long-lasting (stable): straight-line rolling at sufficient speed (i.e., $\theta = \pi/2$, $\dot{\psi}$ 'fast') and spinning about a diameter (i.e., $\theta = \pi/2$, $\dot{\phi}$ 'fast'). We can check our equations by finding the stability conditions for these motions.

For straight line rolling we make the following substitutions and assume α, β, ϕ are so small that we only need to retain first order terms:

$$\dot{\psi} = \dot{\psi}_0 + \beta \tag{25}$$
$$\theta = \pi/2 + \alpha \tag{26}$$
$$\phi = 0 + \phi \tag{27}$$
$$\sin\theta = \cos\alpha = 1 \tag{28}$$
$$\cos\theta = -\sin\alpha = -\alpha \tag{29}$$

Eq. (18) becomes:

$$\begin{pmatrix} (k+1)\,\ddot{\alpha} \\ k\,\ddot{\phi} \\ (2k+1)\,\dot{\beta} \end{pmatrix} = \begin{pmatrix} -\dot{\psi}_0(2k+1)\dot{\phi} + g\alpha \\ 2k\dot{\psi}_0\,\dot{\alpha} \\ 0 \end{pmatrix} \tag{30}$$

From the second component we conclude $\dot{\phi} = 2\dot{\psi}_0\alpha$, which we substitute into the first component:

$$(k+1)\,\ddot{\alpha} = -\left(2(2k+1)\dot{\psi}_0^2 - g\right)\alpha \tag{31}$$

for bounded oscillations of α we must have:

$$\dot{\psi}_0^2 > \frac{g}{2(2k+1)} \tag{32}$$
$$v^2 > \frac{gR}{2(2k+1)} \tag{33}$$

where in the last line we have returned to usually dimensioned quantities.

For spinning about a diameter we make the following substitutions and assume α, β, ψ are so small that we only need to retain first order terms:

$$\dot{\psi} = \dot{\psi} \tag{34}$$
$$\theta = \pi/2 + \alpha \tag{35}$$
$$\dot{\phi} = \dot{\phi}_0 + \beta \tag{36}$$
$$\sin\theta = \cos\alpha = 1 \tag{37}$$
$$\cos\theta = -\sin\alpha = -\alpha \tag{38}$$

Eq. (18) becomes:

$$\begin{pmatrix} (k+1)\,\ddot{\alpha} \\ k\,\dot{\beta} \\ (2k+1)(\ddot{\psi}-\dot{\phi}_0\dot{\alpha}) \end{pmatrix} = \begin{pmatrix} -(\dot{\psi}(2k+1)-\dot{\phi}_0(k+1)\alpha)\dot{\phi}_0+g\alpha) \\ 0 \\ \dot{\alpha}\dot{\phi}_0 \end{pmatrix} \tag{39}$$

From the third component we conclude $(2k+1)\dot{\psi} = 2(k+1)\dot{\phi}_0\alpha$, which we substitute into the first component:

$$(k+1)\,\ddot{\alpha} = -\left((k+1)\dot{\psi}_0^2 - g\right)\alpha \tag{40}$$

from which we conclude for bounded oscillations of α we must have:

$$\dot{\phi}_0^2 \;>\; \frac{g}{(k+1)} \tag{41}$$

Others have disagreed[2].

The usual end-state ($\theta \to 0$) of a spinning coin is striking. The coin's rotation in the inertial frame slows even as the contact point P accelerates in its motion about a nearly stationary CM. In the inertial frame the z component of coin's spin is $\dot{\phi}+\dot{\psi}\cos\theta$. Interestingly if $\dot{\theta}=0$, $\mathbf{v}_{CM} = R(\dot{\psi}+\dot{\phi}\cos\theta)\mathbf{e}_1$, so the relationship between CM motion and apparent spin is mathematically forced in the limit $\theta \to 0$.

If $\dot{\theta}=0$, Eq. (18) reads:

$$\begin{pmatrix} 0 \\ k\,\ddot{\phi}\sin\theta \\ (2k+1)\,\frac{d}{dt}(\dot{\psi}+\dot{\phi}\cos\theta) \end{pmatrix} = \begin{pmatrix} -(\dot{\psi}(2k+1)+\dot{\phi}(k+1)\cos\theta)\dot{\phi}\sin\theta - g\cos\theta \\ 0 \\ 0 \end{pmatrix} \tag{42}$$

From the third component: $\dot{\psi}+\dot{\phi}\cos\theta$=constant, and, based on our observations, we take that constant to be zero. From the second component see that $\dot{\phi}$ and hence $\dot{\psi}$ must separately be constants. Plugging this result into the first component:

$$0 \;=\; -(-(2k+1)+(k+1))\dot{\phi}^2\cos\theta)\sin\theta - g\cos\theta \tag{43}$$

$$0 \;=\; k\dot{\phi}^2\sin\theta - g \tag{44}$$

$$\sqrt{\frac{g}{k\sin\theta}} \;=\; \dot{\phi} \tag{45}$$

so $\dot{\phi}$, which is the angular velocity of the point P, diverges as $\theta \to 0$

Finally we can confirm energy conservation. Dotting $\boldsymbol{\omega}$ with the lhs of Eq. (17) yields

$$\frac{d}{dt}\left[\frac{1}{2}\,\boldsymbol{\omega}\cdot\mathbf{I}\cdot\boldsymbol{\omega}+\frac{1}{2}\,(\omega_1^2+\omega_3^2)\right] = \frac{1}{mR^2}\frac{d}{dt}\left[(\text{rotation KE})+\frac{1}{2}\,mv_{CM}^2\right] \tag{46}$$

so energy conservation requires that $\boldsymbol{\omega}$ dotted with the rhs of Eq. (17) be $-\frac{d}{dt}(mgR\sin\theta) \to -g\cos\theta\dot{\theta}$. Lots of algebra, but *Mathematica* confirms.

The following calculations are for disk: $k=\frac{1}{4}$, $g = 980$ cm/s^2, $R = 1$ cm.

[2]Olsson, M.G. (Am. J. Phys, **40**, 1543 (1972)) asserted that in this situation the horizontal component of **F** was zero, and found $\dot{\phi}_0^2 > \frac{g}{kR}$. McDonald, A.J. derives both results (arXiv:physics/0008227 2001) without comment.

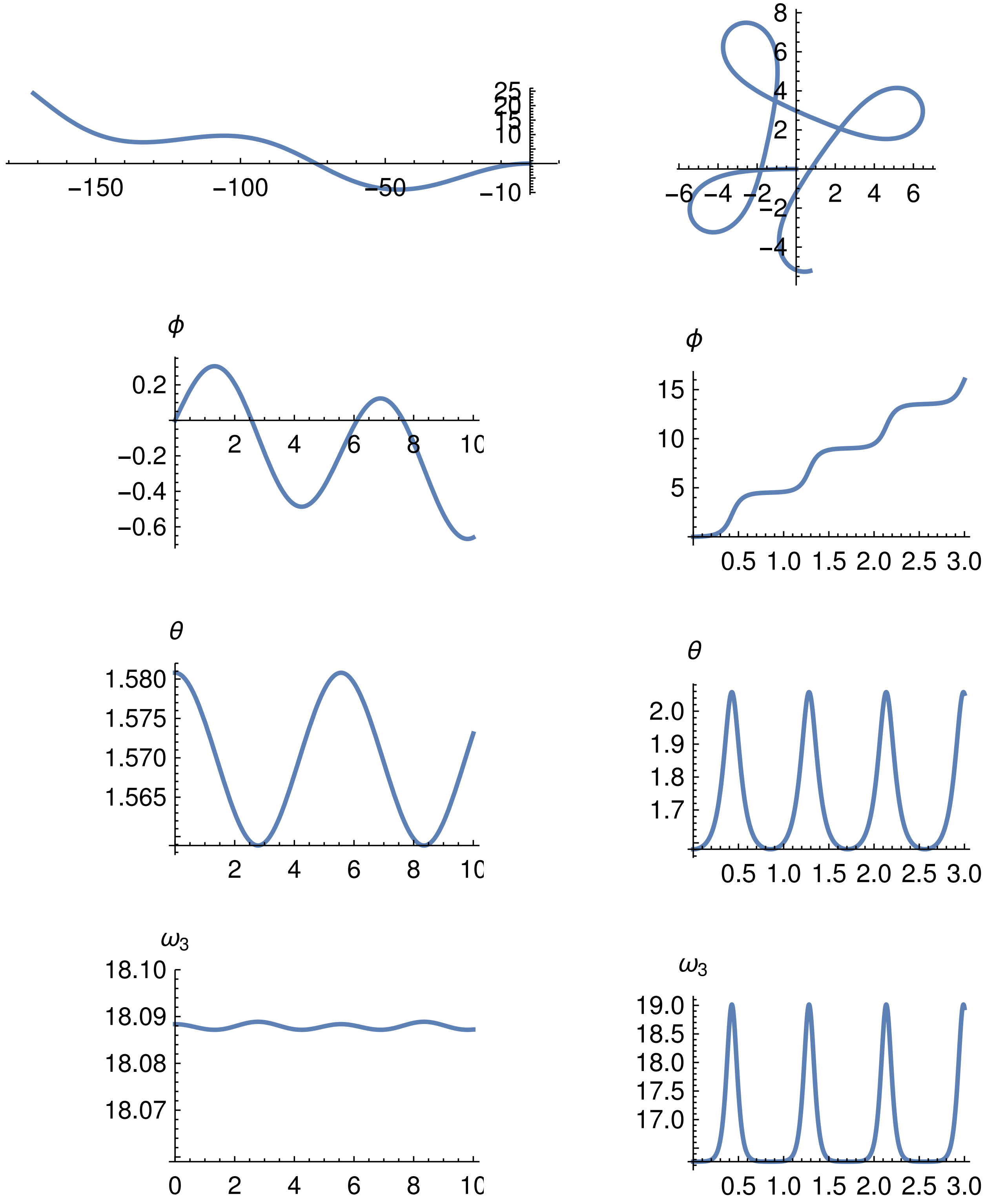

Figure 3: LHS: With $\dot{\psi}$ just 0.1% above the Eq. (32) minimum for stable straight-line motion, and an initial deviation $\alpha = .1$, the disks wobbles above and below $\theta = \pi/2$. While the CM path (top plot) can't be said to be straight, ϕ is generally small, and ω_3 mostly constant. RHS: With $\dot{\psi}$ at 90% of the Eq. (32) minimum, the path is completely wondering, although ϕ shows a consistent increasing trend resulting in a generally clockwise orbit. Note the deep θ 'bows:' the drop in gravitational PE results in a speed up of ω_3.

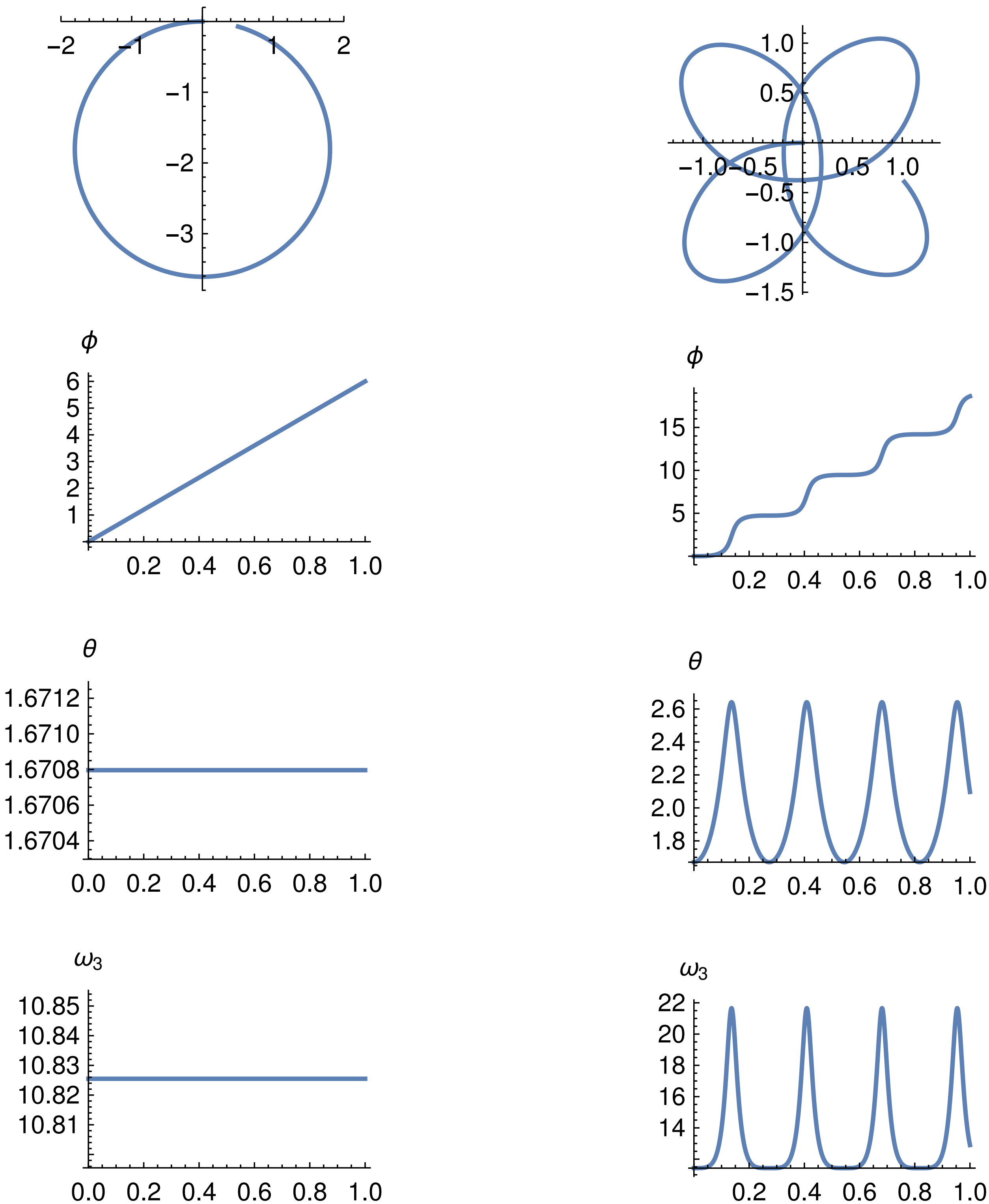

Figure 4: LHS: Starting with exactly the right initial values for circular motion: Eq. (21), circular motion with $\theta, \dot{\phi}, \dot{\psi}$ constant is achieved. Note the very linear ϕ; the disk was started with the proper $\dot{\phi}_0$.
RHS: Starting with $\dot{\phi}_0 = 0$ (check the ϕ plot) and leaving everything else unchanged results in more chaotic motion. Again note deep θ bows when disk changes direction. With most gyroscopes, releasing with $\dot{\phi}_0 = 0$ results in nutation, which is generally more quickly damped then the precession.

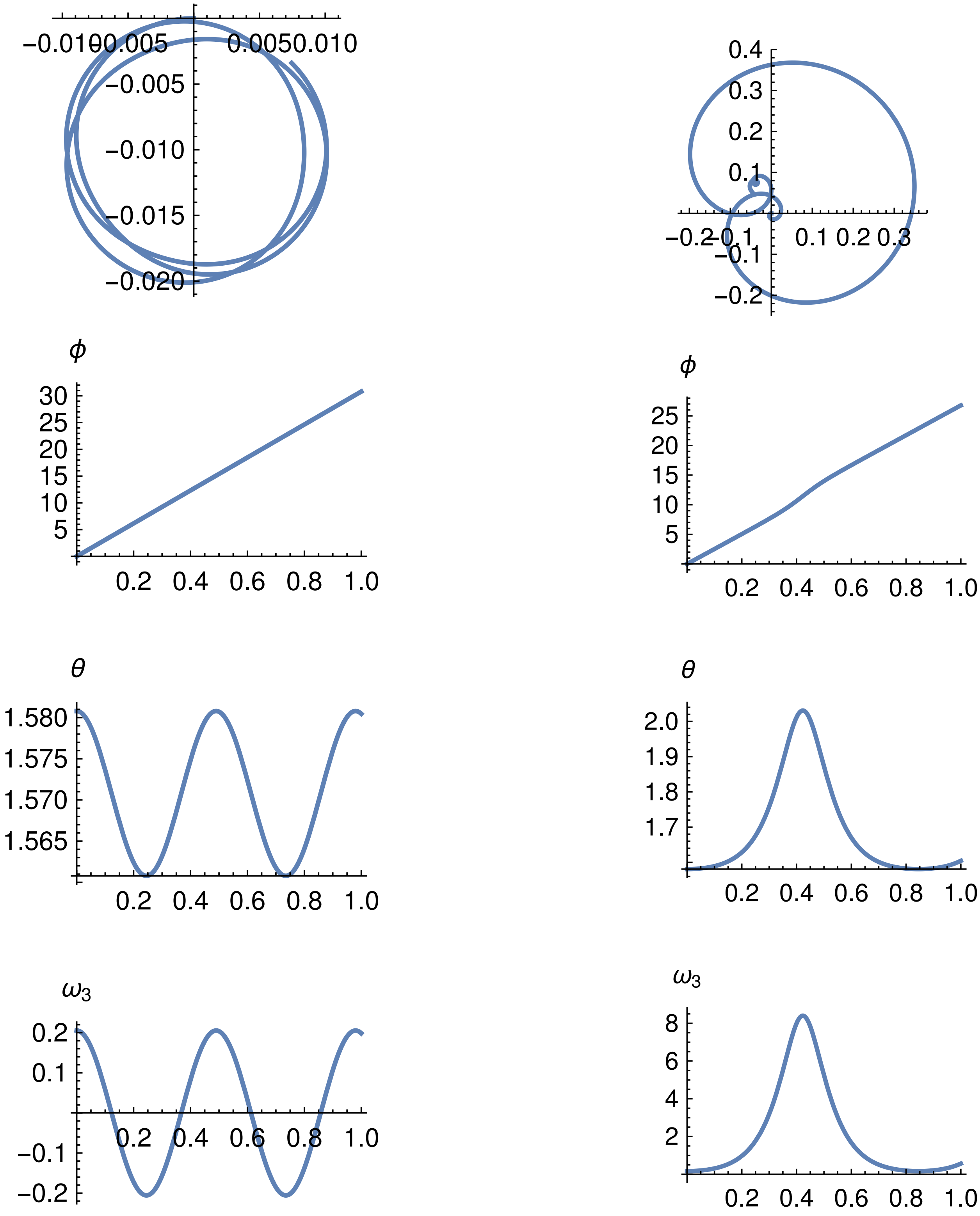

Figure 5: LHS: With $\dot{\phi}$ just 10% above the Eq. (41) minimum for stable vertical diameter spinning and an initial deviation $\alpha = .01$, the disks wobbles above and below $\theta = \pi/2$, ω_3 oscillates in step with θ and the CM path is confined to a small circle.
RHS: With $\dot{\phi}$ at 90% of the Eq. (41) minimum, the path spirals out from the origin and makes a big loop before returning. There is a deep θ bow during the outward sweep accompanied by an increase in ω_3 and $\dot{\phi}$ (note the slight kink in the ϕ line).

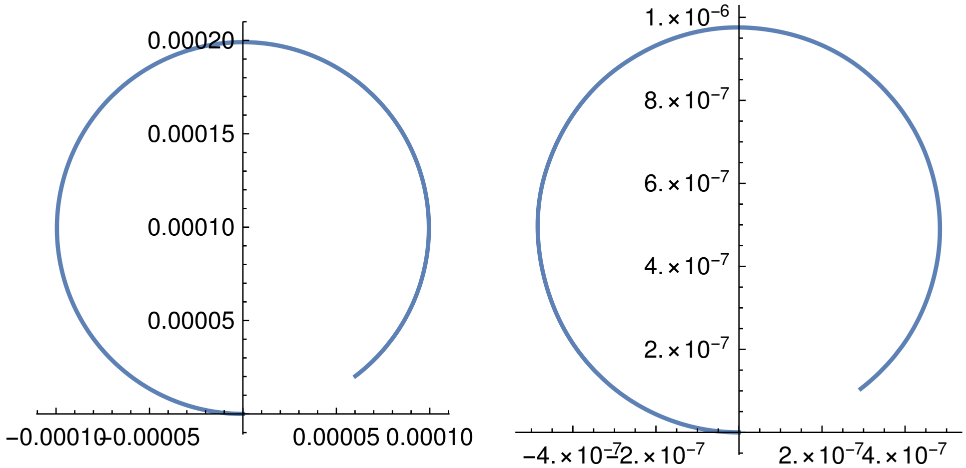

Figure 6: LHS: Spin-down state with no z rotation, but with $v_{CM} \neq 0$: $\dot{\phi} = -625.917$, $\dot{\psi} = 625.948$, $\theta = .01$, $v_{CM}/R = .06$. Note that the resulting CM motion would be hard to detect.
RHS: Spin-down state with $v_{CM} = 0$: $\dot{\phi} = -626.104$, $\dot{\psi} = 626.073$: very similar to the above numbers. The displayed CM motion is just round-off error.

Rolling 3d Objects

There has recently[3] been a great deal of interest in the motion of 3d rings (in contract to 2d hoops). With a 3d object, as shown in Fig. 7, $\mathbf{r}_P = -R\mathbf{e}_2 - h\mathbf{e}_3$, however if tipped further to $\theta > \pi/2$, the other rim becomes the contact point and $\mathbf{r}_P = -R\mathbf{e}_2 + h\mathbf{e}_3$. For θ near $\pi/2$ ($|\alpha| < \tan^{-1}(h/R)$), the gravitational torque on the CM about P will produce a stable, non-rotating equilibrium of the coin teetering back and forth. Additionally tipping from one rim to the other is inconsistent with rolling without slipping unless $\boldsymbol{\omega}$ is exclusively in the $\mathbf{e}_3$ direction. Our equations of motion will not accurately describe motions that send θ through $\pi/2$. We find it convenient to consider only motions with $\theta < \pi/2$.

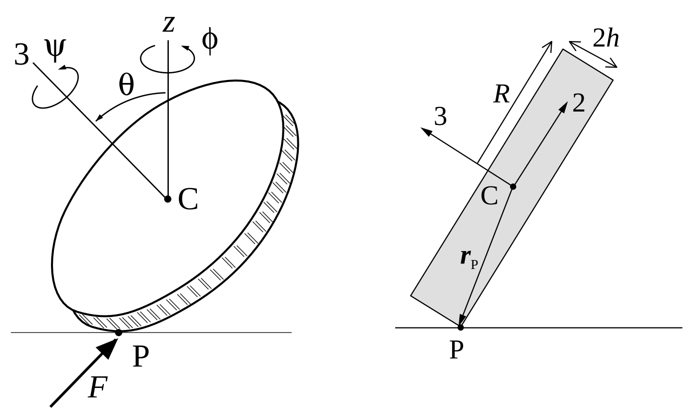

Figure 7: Rolling coins, since they have a thickness $2h$, only approximate rolling disks. There are additional terms in the equations of motion because $\mathbf{r}_P$ picks up a component in the direction of $\mathbf{e}_3$. Note: in the RHS diagram $\mathbf{e}_1$ points out of the page.

[3]M. A. Jalali, M. S. Sarebangholi, and M.-R. Alam, Phys. Rev. E **92**, 032913 (2015) (arXiv:1412.1852v2 [physics.class-ph])
A. V. Borisov, A. A. Kilin, Y. L. Karavaev; *Comments on the paper "Terminal retrograde turn of rolling rings"* (arXiv:1611.02957v1 [physics.class-ph])

Starting from Eq. (10) above:

$$\mathbf{I}\cdot\dot{\boldsymbol{\omega}} + \mathbf{r}_P \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}_P) = -\boldsymbol{\Omega} \times (\mathbf{I}\cdot\boldsymbol{\omega}) + \mathbf{r}_P \times (\boldsymbol{\Omega} \times (\mathbf{r}_P \times \boldsymbol{\omega}) - \mathbf{g}) \quad (47)$$

$$\mathbf{I}\cdot\dot{\boldsymbol{\omega}} + r_P^2\dot{\boldsymbol{\omega}} + (R\dot{\omega}_2 + h\dot{\omega}_3)\mathbf{r}_P = -\boldsymbol{\Omega} \times (\mathbf{I}\cdot\boldsymbol{\omega}) + \mathbf{r}_P \times ((R\Omega_2 + h\Omega_3)\boldsymbol{\omega} - \mathbf{g}) \quad (48)$$

Renaming $\mathbf{I} \leftarrow \mathbf{I}/R^2$, $g \leftarrow g/R$, and $h \leftarrow h/R$ gives:

$$\mathbf{I}\cdot\dot{\boldsymbol{\omega}} + (1+h^2)\dot{\boldsymbol{\omega}} - (\dot{\omega}_2 + h\dot{\omega}_3)(\mathbf{e}_2 + h\mathbf{e}_3) = \\ -\boldsymbol{\Omega} \times (\mathbf{I}\cdot\boldsymbol{\omega}) - (\Omega_2 + h\Omega_3)((\mathbf{e}_2 + h\mathbf{e}_3) \times \boldsymbol{\omega}) - (\cos\theta - h\sin\theta)g\mathbf{e}_1 \quad (49)$$

$$\begin{pmatrix} (I_1+1+h^2)\,\dot{\omega}_1 \\ (I_1+h^2)\,\dot{\omega}_2 - h\dot{\omega}_3 \\ (I_3+1)\,\dot{\omega}_3 - h\dot{\omega}_2 \end{pmatrix} = \begin{pmatrix} (I_1+1+h^2)\,\ddot{\theta} \\ (I_1+h^2)\,(\ddot{\phi}\sin\theta + \dot{\phi}\cos\theta\dot{\theta}) - h(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\sin\theta\dot{\theta}) \\ (I_3+1)\,(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\sin\theta\dot{\theta}) - h(\ddot{\phi}\sin\theta + \dot{\phi}\cos\theta\dot{\theta}) \end{pmatrix} = \\ \begin{pmatrix} -\dot{\phi}\dot{\psi}((1+I_3)\sin\theta + h\cos\theta) - \dot{\phi}^2(\sin\theta\cos\theta(1+I_3-I_1-h^2) + h\cos 2\theta) - g(\cos\theta - h\sin\theta) \\ \dot{\theta}(\dot{\psi}I_3 + \dot{\phi}((I_3 - I_1 - h^2)\cos\theta - h\sin\theta)) \\ \dot{\phi}\dot{\theta}(\sin\theta + h\cos\theta) \end{pmatrix} \quad (50)$$

Evidently the 3d equations of motion are much more complicated than the 2d case. Some checks are in order.

Much as with the disk, the first component of Eq. (50) is substantially tested by considering a ring whose the contact point P traces a circle of radius ρ. The choice of $\theta < \pi/2$ results in some unexpected signs. In Fig. 8, I assume $\dot{\psi} > 0$, so the ring is rolling into the page and, just as in Fig. 7, the 1-axis is out of the page (so $\theta > 0$) and the 2-axis, perpendicular to both, points towards the 2 o'clock position. Orbiting the displayed circle center requires the ring to turn towards the right, which is $\dot{\phi} < 0$.

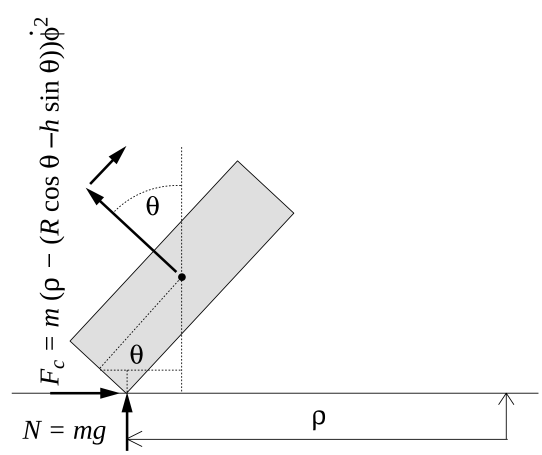

Figure 8: A ring moves in a circle (radius ρ) at constant θ. The forces/torques are designed to rotated the CM and precess $\mathbf{L}$ at exactly the same constant rate: $|\dot{\phi}|$. (For convenience the figure has assumed $L_2 > 0$ which would not actually be the case.)

The torques from the contact precess the horizontal component of the angular momentum ($L_\perp$) at exactly the same angular rate as CM moves: $|\dot{\phi}|$. Using Eq. (16) and the geometry displayed in Fig. 8:

$$L_\perp = I_3(\dot{\psi} + \dot{\phi}\cos\theta)\sin\theta - I_1\dot{\phi}\sin\theta\cos\theta \quad (51)$$

The net torque (into of the page) is:

$$\Gamma = mg\,(R\cos\theta - h\sin\theta) - m\,(\rho - (R\cos\theta - h\sin\theta))\,\dot{\phi}^2\,(R\sin\theta + h\cos\theta) \quad (52)$$

$$\rightarrow g(\cos\theta - h\sin\theta) - \Big(-\dot{\psi} - \dot{\phi}(\cos\theta - h\sin\theta)\Big)\,\dot{\phi}\,(\sin\theta + h\cos\theta) \quad (53)$$

where we have used: $\dot\psi = -\rho\dot\phi/R$. Setting $L_\perp|\dot\phi| = \Gamma$ produces:

$$-\Big(I_3(\dot\psi + \dot\phi\cos\theta)\sin\theta - I_1\dot\phi\sin\theta\cos\theta\Big)\,\dot\phi = \\ g(\cos\theta - h\sin\theta) - \Big(-\dot\psi - \dot\phi(\cos\theta - h\sin\theta)\Big)\,\dot\phi(\sin\theta + h\cos\theta) \quad (54)$$

$$-\dot\psi\dot\phi\,((I_3+1)\sin\theta + h\cos\theta) - \dot\phi^2\,((\cos\theta - h\sin\theta)\,(\sin\theta + h\cos\theta) + (I_3 - I_1)\cos\theta\sin\theta) \\ - g(\cos\theta - h\sin\theta) = 0 \quad (55)$$

$$-\dot\psi\dot\phi\,((I_3+1)\sin\theta + h\cos\theta) - \dot\phi^2\,\big(h(\cos^2\theta - \sin^2\theta) + (I_3 - I_1 + 1 - h^2)\cos\theta\sin\theta\big) \\ - g(\cos\theta - h\sin\theta) = 0 \quad (56)$$

$$-\dot\psi\dot\phi\,((I_3+1)\sin\theta + h\cos\theta) - \dot\phi^2\,\big(h\cos 2\theta + (I_3 - I_1 + 1 - h^2)\cos\theta\sin\theta\big) \\ - g(\cos\theta - h\sin\theta) = 0 \quad (57)$$

Compare to component 1 of Eq. (50). Note that the other components are also satisfied as $\dot\theta = 0$ then results in $\ddot\phi = \ddot\psi = 0$. Reversing ourselves, we replace $\dot\psi = -\rho\dot\phi/R$ and using our usual rescaling $\rho \leftarrow \rho/R$ we can find the $\dot\phi$ for any such circular motion:

$$\dot\phi^2\left[\rho\,((I_3+1)\sin\theta + h\cos\theta) - \big(h\cos 2\theta + (I_3 - I_1 + 1 - h^2)\cos\theta\sin\theta\big)\right] = \\ g(\cos\theta - h\sin\theta) \quad (58)$$

If $\rho = (\cos\theta - h\sin\theta)$ the CM is at rest, exactly at the center of the circle.

Again any combination of $\dot\psi =$ constant, $\dot\phi =$ constant that makes the first component of Eq. (50) zero (allowing $\dot\theta = 0$) is a solution to these equations of motion. We examine this in the limit $\theta \to 0$, by making a Taylors series expansion of first component of Eq. (50):

$$-g - h\left[\dot\phi(\dot\phi + \dot\psi)\right] + \left(gh + \dot\phi^2(h^2 + I_1) - \left[\dot\phi(\dot\phi + \dot\psi)\right]\right)\theta \quad (59)$$

Defining $Y \equiv \left[\dot\phi(\dot\phi + \dot\psi)\right]$ and seeking when the 1 component is zero results:

$$-g(1-h) + \dot\phi^2(h^2 + I_1)\theta = (h + (1 + I_3)\theta)\,Y \quad (60)$$

The end state situation of $\theta \to 0$ with no rotation requires $(\dot\phi + \dot\psi) \to 0$. If Y is small compared to g (basically because $|\dot\phi + \dot\psi| \ll |\dot\phi|$) we obtain the equivalent of Eq. (45):

$$\dot\phi = \sqrt{\frac{g(1-h)}{(h^2 + I_1)\,\theta}} \quad (61)$$

which shows the divergence of $\dot\phi$ as $\theta \to 0$.

We can examine the straight-line ($\dot\phi = 0$) rolling solution with the CM directly above the contact point (θ nearly $\pi/2$): $\cos\theta - h\sin\theta = 0$ or $\cot^{-1}(h) = \theta$. Clearly this is a solution for any $\dot\psi$. Algebraically the stability of this solution is a mess, so we investigate it numerically.

The 'diameter spin' solutions ($\dot{\phi} \neq 0$, $\dot{\psi} = 0$, $\dot{\theta} = 0$) require finding the angle θ_0 which is a root of the first component of Eq. (50):

$$-\dot{\phi}^2(\sin\theta\cos\theta(1 + I_3 - I_1 - h^2) + h\cos 2\theta) - g(\cos\theta - h\sin\theta) \tag{62}$$

θ_0 ranges between $\cot^{-1}(h)$ for slow spin, and $\tan^{-1}(-2h/(1 + I_3 - I_1 + h^2))/2$ for fast spin (between 1.16 and 1.32 for our example). Algebraically the stability of this solution is a mess, so we investigate it numerically.

The following calculations are based on an in-hand ring: $R = 7.32/2$ cm, $h = 3.18/2$ cm, $w = .5$ cm. This results in scaled quantities $I_3 = (1 + (1 - w/R)^2)/2 = 0.8727$, $I_1 = (3(1 + (1 - w/R)^2) + (2h/R)^2)/12 = 0.4993$, $h \leftarrow h/R = 0.4344$. We keep $g/R = 980$ cm/s^2, i.e., our mathematical ring is proportional to the above but with $R = 1$ cm. Time is in seconds (unless otherwise stated it has not been scaled).

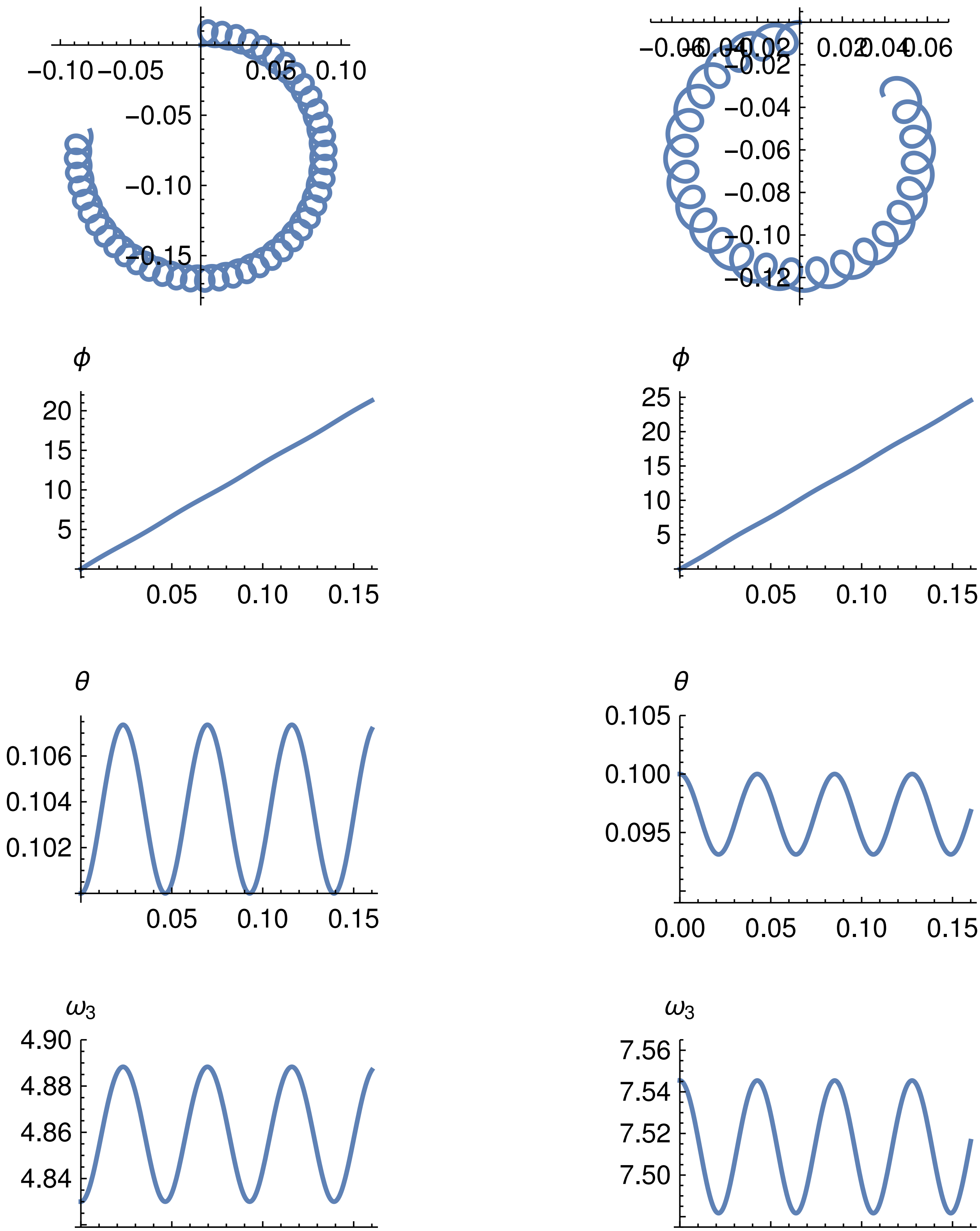

Figure 9: Following the procedure suggested by Eq. (58), we can find the initial conditions for static ($\dot{\psi}$ = constant, $\dot{\phi}$ = constant, θ = constant), circular motion with $\theta_0 = .1$
LHS: With $\dot{\psi}_0$ just 1% above that for stable circular motion: see clockwise CM motion.
RHS: With $\dot{\psi}_0$ just 1% below that for stable circular motion: see counter-clockwise CM motion.

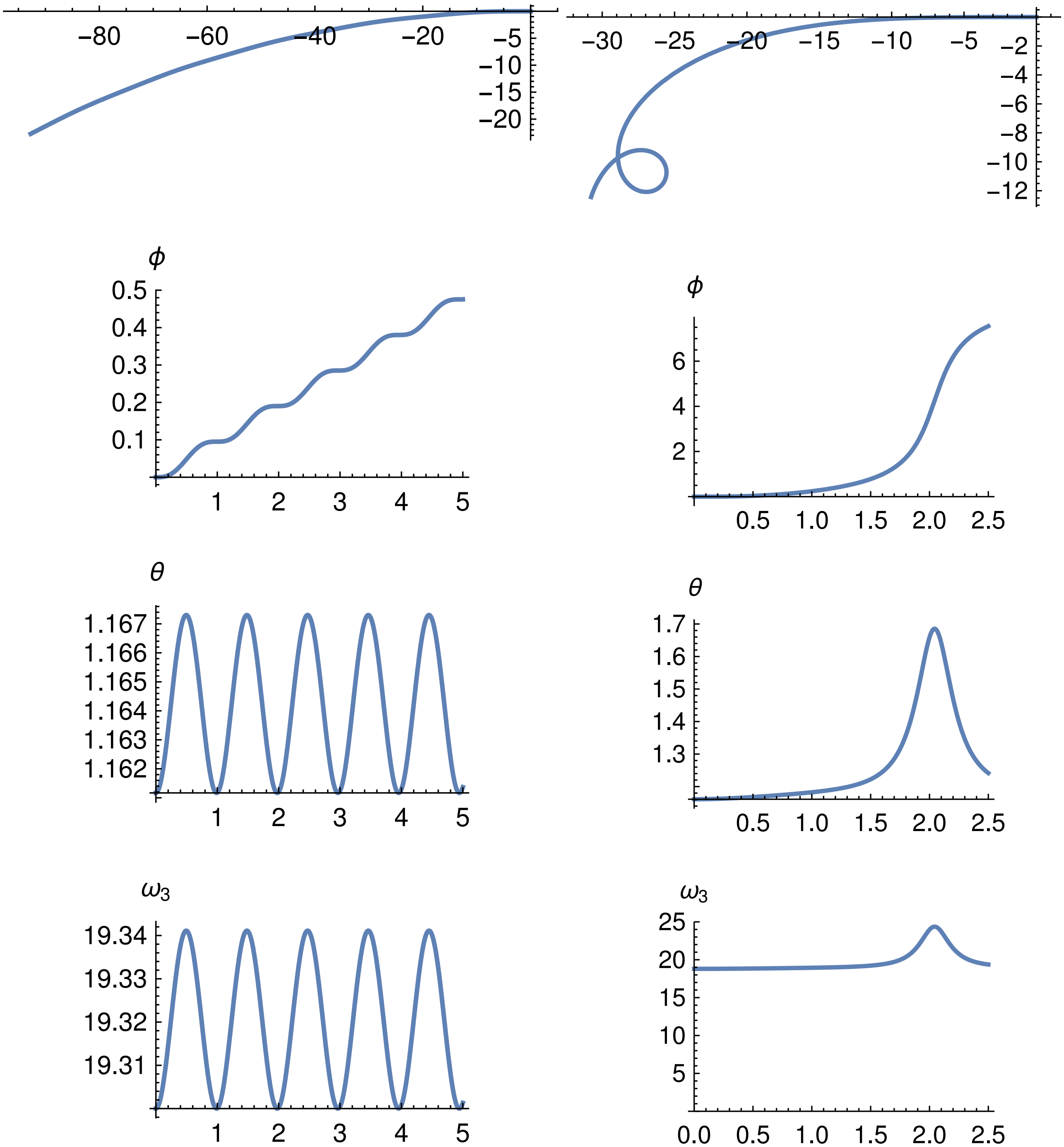

Figure 10: Straight line motion solutions, where the ring is started with its CM almost ($\Delta\theta = .0002$) directly above the contact ($\theta = \cot^{-1} h = 1.161$).
LHS: With $\dot{\psi}_0 = 19.3$, θ oscillates, but just on one side of equilibrium. As a result the path slowly deviates from a straight line.
RHS: With $\dot{\psi}_0 = 18.8$, θ quickly exceeds the limit of $\pi/2$.

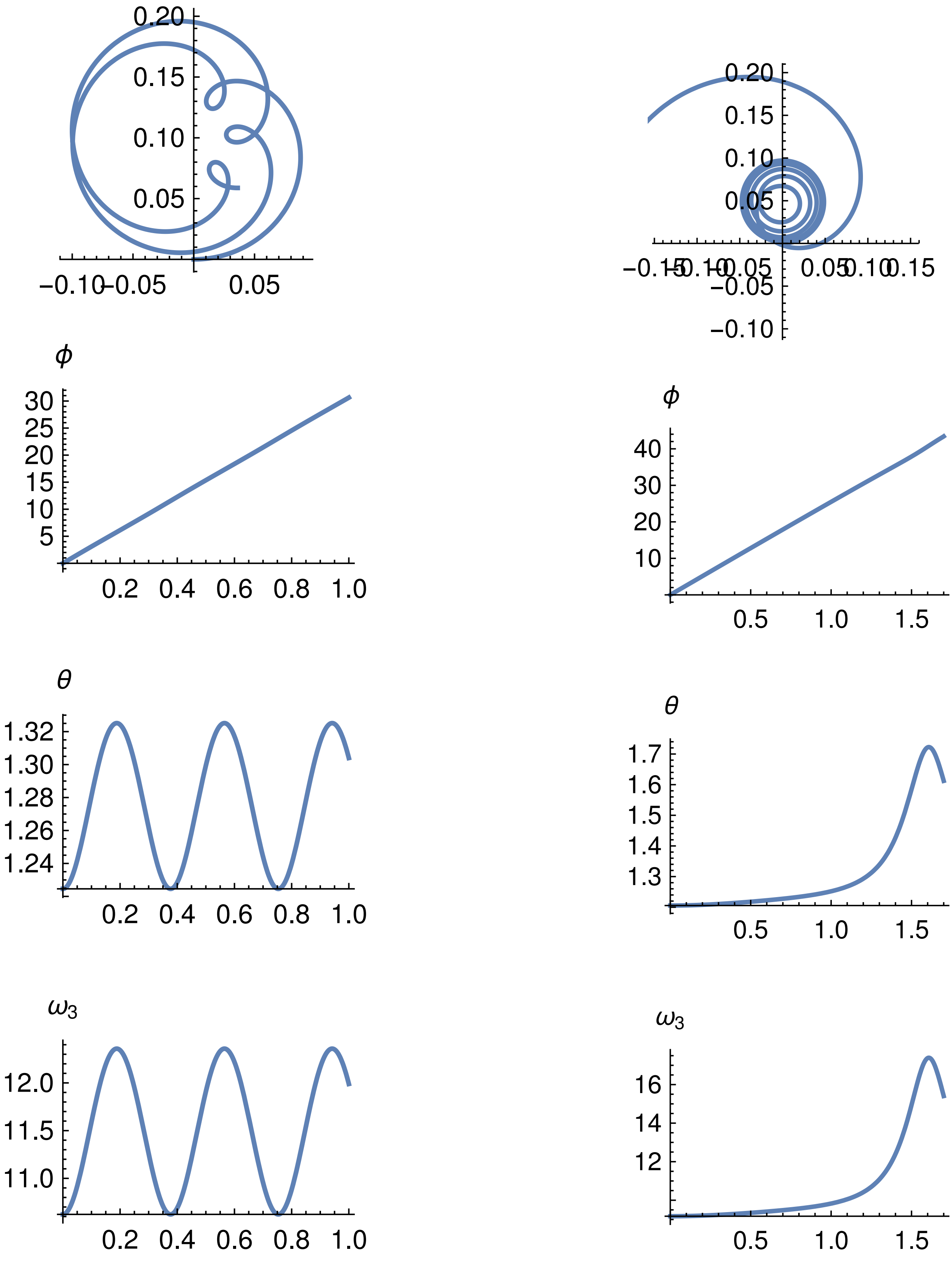

Figure 11: 'Diameter spin' solutions, where the ring is started near its equilibrium angle θ_0, with $\dot{\psi}_0 = 0$ and $\dot{\phi}_0$ 'fast'
LHS: With $\dot{\phi}_0 = \sqrt{g/R} = 31.3$, $\theta_0 = 1.225$: θ oscillates.
RHS: With $\dot{\phi}_0 = 25.7$, $\theta_0 = 1.205$: θ quickly exceeds the limit of $\pi/2$.

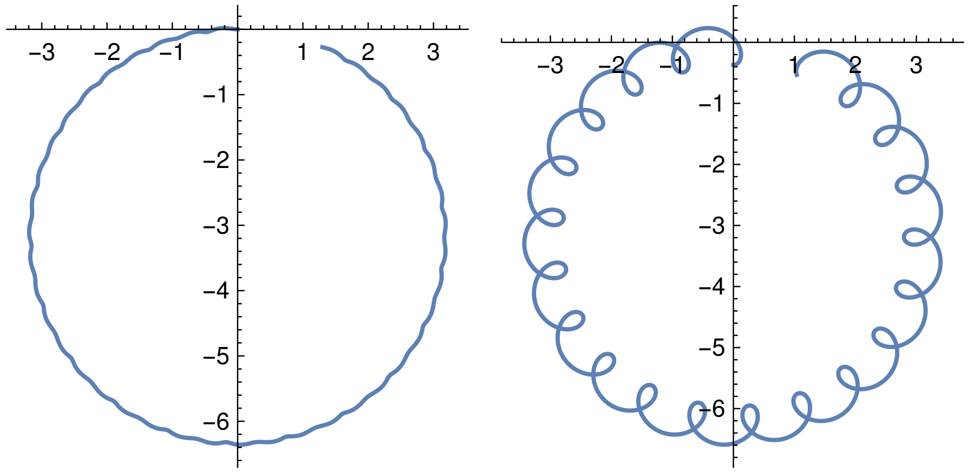

Figure 12: Following FIG. 1 from Jalali, et al. we consider initial conditions: $\theta_0 = \pi/2 - .55$, $\dot{\phi}_0 = 4.5\sqrt{g/R}$, $\dot{\psi}_0 = 0$. It is important to recognize that workers commonly plot the motion of the point A: $\mathbf{r}_A = (0, 0, h)$, i.e., a point on the top level of the ring, directly above the CM. The CM may have rather different motion! Also note that workers may use dimensionless time: $t \leftarrow t\sqrt{g/R}$ which effectively sets $g = 1$ in the equations of motion.
LHS: CM motion.
RHS: motion of the point A.

Much of the interest in the classical dynamics of rolling rings was caused by the observation that, in orbit decay, rolling disks consistently show one direction of circular CM motion, whereas rings, in a single run, show both motions like Fig. 9 LHS and Fig. 9 RHS, that is, the end state motion (say clockwise) will be preceded by a counter-clockwise phase. We are now talking about the evolution of a non-conservative system. In some sense it's no surprise that forces not included in our mathematical model could cause the small shift that's required to shift from Fig. 9 LHS to Fig. 9 RHS. The original paper purposed that air drag caused the difference, and that the presence (ring) or absence (disk) of a central hole produced different drag coefficients. Experiments have ruled this out: use light weight material to fill the ring's hole without much changing $\mathbf{I}$ (air drag should depend on object shape not material make-up) or do the experiment at low pressure. Another force not in our model is rolling friction. It is important to recognize that rolling friction cannot be the result of a single force applied at a single contact point. Rolling friction is much more complex than the simple sliding friction discussed in intro physics: it depends on the material squish of wheel and surface when they come into contact. As such it is a problem in continuum mechanics (material flow) just as air drag is. If the air at the trailing end of a moving object moved analogously to the air at the leading end, the non-viscous air drag would be zero (d'Alembert's paradox). Similarly if the leading edge force as the surface/wheel is compressed were the same as the trailing edge force as the material is released, there would be no rolling resistance. A problem with the rolling friction explanation is that it's hard to see how the circular rim of a disk could produce any different rolling friction force/torque than the circular rim of a ring. But of course the effect of that additional force gets filtered through the equations of motion, and we've seen that not much change is required. Workers have reported that various very simple models of rolling friction (akin to the approximate v^2 air drag force) can account for the reversing circular motion. Indeed we could take the original paper's mathematical model

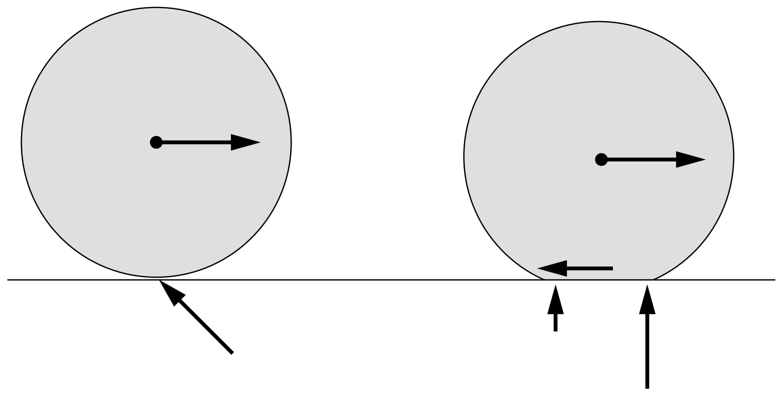

Figure 13: A wheel rolls to the right. If, as in the LHS, we limit contact to a single point, no single force is consistent with drag. The displayed force vector would slow the CM, but as a torque it would accelerate the rotation. This is not consistent! In the RHS the contact is a footprint not a single point. The forces involved in compression at the leading edge are larger than the same forces at the trailing edge. The result is a decelerating torque (and a net normal force). Add a retarding force (with a corresponding torque less than that due to the material compression), and we have a force system that can simultaneously slow the CM and the rotation.

of air drag, and just call it rolling friction.

Hurricane Balls

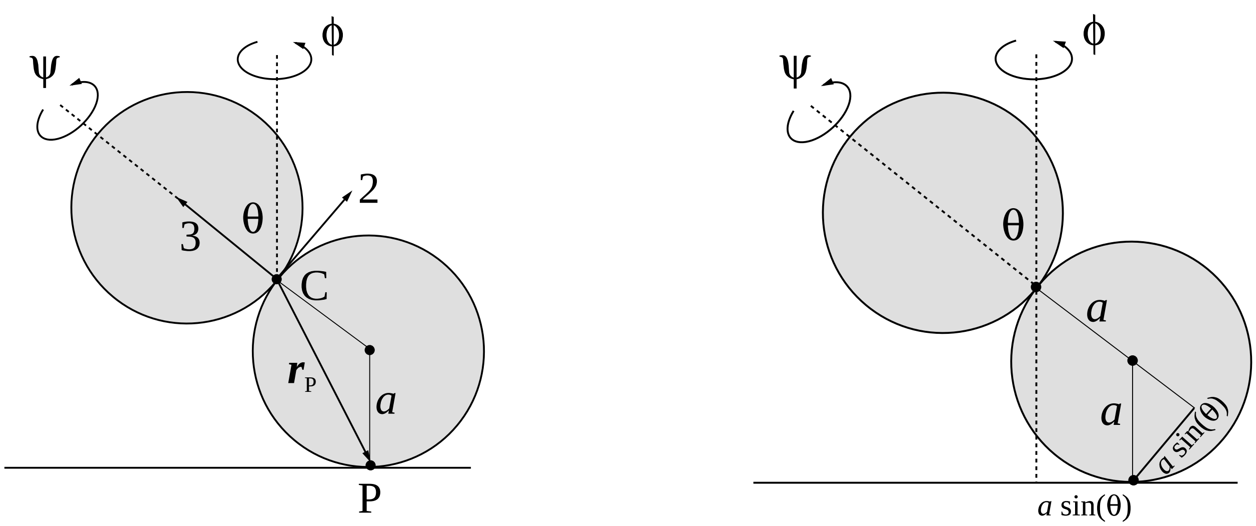

Figure 14: Hurricane Balls consist of two spheres rigidly attached at a point.

Hurricane Balls are another symmetric toy that rolls without slipping. It has some significant differences form other objects that we've considered. First $\mathbf{r}_P$ is not a constant:

$$\mathbf{r}_P = a(0, -\sin\theta, -1 - \cos\theta) \tag{63}$$

Second, it is 'prolate', i.e., $I_1 > I_3$:

$$\mathbf{I} = \frac{2}{5}\, ma^2 \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2ma^2 \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{64}$$

where m is the mass of one ball. Let's begin with some simple consequences. If the CM remains fixed (hence θ fixed), the contact point P will make a circle of radius $a \sin\theta$ in the plane. That contact will also inscribe a circle on the lower ball which also has radius $a \sin\theta$. As the object rotates, the arc length inscribed on the plane and on the lower ball must be the same; same arc length and same radius means same angle, so $\Delta\psi = \Delta\psi$. We can reach the same conclusion with more algebra by using out equation for $\mathbf{v}_{CM}$:

$$\begin{aligned} \mathbf{v}_{CM} = -\boldsymbol{\omega} \times \mathbf{r}_P &= -a(\dot\theta, \dot\phi\cos\theta, \dot\phi\sin\theta + \dot\psi) \times (0, -\sin\theta, -1 - cos\theta) && (65) \\ &= a\left((\dot\phi - \dot\psi)\sin\theta, -\dot\theta(1+\cos\theta), \dot\theta\sin\theta\right) && (66) \end{aligned}$$

So if $\dot\phi = \dot\psi$ and $\dot\theta = 0$ the CM is at rest, but if $\dot\theta \neq 0$ the CM will have both vertical (since $z = a(1 + \cos\theta)$) and horizontal components of velocity. $\dot\phi = \dot\psi$ =constant is inconsistent with $\dot\theta \neq 0$ (see the third component of Eq. (80-81) below). Solving the actual equations of motion will show that generally if $\dot\theta \neq 0$, $\dot\phi \neq \dot\psi$. 'Lagrangians' that employ $\dot\phi = \dot\psi$ as a constraint and include $\dot\theta$ are not solving the actual physics.

We can pursue the familiar game of finding solutions where P moves in a circle. Initially we seek solutions with the CM at rest, so there is no horizontal contact force. Following Fig. 8, note the additional differences: P is to the right of the CM so the torque due to the normal force is out of the page, which is as expected as $\dot\phi = \dot\psi > 0$. $L_\perp$ is exactly as in Eq. (51); the torque due to the normal force is

$$\Gamma \;=\; 2mga\sin\theta \tag{67}$$

Setting $L_\perp\dot\phi = \Gamma$, dividing out the scale factor $2ma^2$, and using $g \leftarrow g/a$ results

$$\begin{aligned} L_\perp\dot\phi = \dot\phi\left[I_3(\dot\psi + \dot\phi\cos\theta)\sin\theta - I_1\dot\phi\sin\theta\cos\theta\right] &= 2mga\sin\theta && (68) \\ \dot\phi^2\left[I_3(1+\cos\theta) - I_1\cos\theta\right] &= g && (69) \\ \left[I_3 - (I_1 - I_3)\cos\theta\right] &= \frac{g}{\dot\phi^2} && (70) \\ \frac{2}{5} - \cos\theta &= \frac{g}{\dot\phi^2} && (71) \\ \frac{2}{5} - \frac{g}{\dot\phi^2} &= \cos\theta && (72) \end{aligned}$$

Our starting point for the full equations of motion is Eq. (8), but with an additional term because $\mathbf{r}_P$ is not constant:

$$\mathbf{I}\cdot\dot{\boldsymbol{\omega}} + \boldsymbol{\Omega} \times \mathbf{L} = 2m\mathbf{r}_P \times (-\dot{\boldsymbol{\omega}} \times \mathbf{r}_P - \boldsymbol{\omega} \times \dot{\mathbf{r}}_P + \boldsymbol{\Omega} \times \mathbf{v}_C - \mathbf{g}) \tag{73}$$

As usual I'll put the ω time derivatives on the LHS:

$$\mathbf{I}\cdot\dot{\boldsymbol{\omega}} + 2m\mathbf{r}_P \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}_P) = -\boldsymbol{\Omega} \times \mathbf{L} + 2m\mathbf{r}_P \times (-\boldsymbol{\omega} \times \dot{\mathbf{r}}_P + \boldsymbol{\Omega} \times (\mathbf{r}_P \times \boldsymbol{\omega}) - \mathbf{g}) \tag{74}$$

You can quickly check energy conservation in this form, by dotting with $\boldsymbol{\omega}$:

$$\begin{aligned}
\boldsymbol{\omega}\cdot\mathbf{I}\cdot\dot{\boldsymbol{\omega}} &= \frac{d}{dt}(\text{rotational KE}) && (75)\\
2m\,\boldsymbol{\omega}\cdot\left[\mathbf{r}_P\times\frac{d}{dt}\left(\boldsymbol{\omega}\times\mathbf{r}_P\right)\right] &= 2m\ \left(\boldsymbol{\omega}\times\mathbf{r}_P\right)\cdot\frac{d}{dt}\left(\boldsymbol{\omega}\times\mathbf{r}_P\right)=\frac{d}{dt}(\text{CM KE}) && (76)\\
-\boldsymbol{\omega}\cdot\boldsymbol{\Omega}\times\mathbf{L} &\sim \boldsymbol{\omega}\cdot(\boldsymbol{\omega}\&\mathbf{e}_3)\times(\boldsymbol{\omega}\&\mathbf{e}_3)\sim\boldsymbol{\omega}\cdot(\boldsymbol{\omega}\times\mathbf{e}_3)=0 && (77)\\
2m\,\boldsymbol{\omega}\cdot\mathbf{r}_P\times(\boldsymbol{\Omega}\times(\mathbf{r}_P\times\boldsymbol{\omega})) &= 2m\,\boldsymbol{\omega}\cdot\mathbf{r}_P\times(-(\boldsymbol{\Omega}\cdot\mathbf{r}_P)\boldsymbol{\omega})=0 && (78)\\
-2m\,\boldsymbol{\omega}\cdot\mathbf{r}_P\times\mathbf{g} &= 2mg\boldsymbol{\omega}\cdot\mathbf{e}_1\sin\theta=2mg\sin\theta\,\dot{\theta}=-\frac{d}{dt}\left(2mg\cos\theta\right) && (79)
\end{aligned}$$

Mathematica finds the RHS of this is:

$$\begin{pmatrix}\sin\theta\left(g+\dot{\phi}^2((I_1-I_3+1)\cos\theta+1)-\dot{\phi}\dot{\psi}(I_3+\cos\theta+1)+\dot{\theta}^2\right)\\ \dot{\theta}\left(\dot{\psi}\left(I_3+\cos^2\theta+\cos\theta\right)-\dot{\phi}\cos\theta(I_1-I_3+\cos\theta+1)\right)\\ \dot{\theta}\sin\theta\cos\theta(\dot{\phi}-\dot{\psi})\end{pmatrix} \quad (80)$$

You can check that $\dot{\psi}=\dot{\phi}$, $\dot{\theta}=0$, and $\cos\theta=2/5-g/\dot{\phi}^2$ gives zero for the above. *Mathematica* finds the LHS:

$$\begin{pmatrix}\ddot{\theta}(I_1+2\cos\theta+2)\\ \sin\theta\left(\ddot{\phi}(I_1+\cos\theta+1)-\ddot{\psi}(\cos\theta+1)\right)+\dot{\phi}\dot{\theta}(I_1\cos\theta+(1+\cos\theta)^2)\\ \ddot{\phi}(I_3\cos\theta+\cos^2\theta-1)-\dot{\phi}\dot{\theta}\sin\theta(I_3+\cos\theta+1)+\ddot{\psi}(I_3-\cos^2\theta+1)\end{pmatrix} \quad (81)$$

We can now look for circular CM motions with $\dot{\psi}=\rho\dot{\phi}$. Referring to Fig. 14, $\rho>1$ corresponds to the CM to the right of the circle center and requires a centripetal force to the left whose torque opposes the normal force's torque, and the reverse for $\rho<1$. Seeking the first component of Eq. (80) to be zero:

$$\cos\theta=\frac{\left(\frac{7}{5}\,\rho-1\right)-\frac{g}{\dot{\phi}^2}}{2-\rho} \quad (82)$$

So as ρ ranges from $\frac{5}{4}$ to 1, high-spin solutions with $0<\theta<\cos^{-1}\left(\frac{2}{5}\right)$ are found but they have (fairly small) CM motion. For $\rho>\frac{5}{4}$, we can find constant θ, circular CM motions with finite $\dot{\phi}$ and $\theta\approx 0$. The solution with the smallest $\dot{\phi}+\dot{\psi}\approx 1.94\sqrt{g}$ occurs at $\rho=3.5$.

Below I contrast the behavior of the rolling motion of the hurricane balls with the behavior of a gyroscope made of hurricane balls with the bottom ball bottom ($\mathbf{r}_B=(0,0,-2a)$) held fixed. From previous results we have the rotational KE:

$$T_{rot}=\frac{1}{2}\,I_1\left(\dot{\phi}^2\sin^2(\theta)+\dot{\theta}^2\right)+\frac{1}{2}\,I_3\left(\dot{\phi}\cos(\theta)+\dot{\psi}\right)^2 \quad (83)$$

KE of the CM

$$T_{CM}=\frac{1}{2}\,2m(2a)^2\left(\dot{\phi}^2\sin^2(\theta)+\dot{\theta}^2\right) \quad (84)$$

and PE of the CM:

$$V=2m2ag\ \cos\theta \quad (85)$$

Of course we'll want to scale out the usual $2ma^2$. This can be manipulated into a (perhaps) familiar form by combining T_{rot} and T_{CM} with results $\tilde{I}_1 = I_1 + 4$ (this is basically the parallel axis theorem) and $\tilde{g} = 2g/a$.

$$L = \frac{1}{2}\,\tilde{I}_1\left(\dot{\phi}^2\sin^2(\theta) + \dot{\theta}^2\right) + \frac{1}{2}\,I_3\left(\dot{\phi}\cos(\theta) + \dot{\psi}\right)^2 - \tilde{g}\cos\theta \tag{86}$$

Dropping the tilde for notational ease, note that ψ and ϕ are ignorable with canonical momenta

$$p_\psi = \frac{\partial T}{\partial\dot{\psi}} = I_3\left(\dot{\phi}\cos(\theta) + \dot{\psi}\right) \tag{87}$$

$$p_\phi = \frac{\partial T}{\partial\dot{\phi}} = I_3\cos(\theta)\left(\dot{\phi}\cos(\theta) + \dot{\psi}\right) + I_1\,\dot{\phi}\sin^2(\theta) = p_\psi\cos(\theta) + I_1\,\dot{\phi}\sin^2(\theta) \tag{88}$$

which gives us the energy:

$$E = \frac{1}{2}\,I_1\,\dot{\theta}^2 + \frac{\left(p_\phi - p_\psi\cos(\theta)\right)^2}{2I_1\sin^2(\theta)} + \frac{p_\psi^2}{2I_3} + g\cos\theta = \frac{1}{2}\,I_1\,\dot{\theta}^2 + V(\theta) \tag{89}$$

where V is

$$V = \frac{\left(p_\phi - p_\psi\cos(\theta)\right)^2}{2I_1\sin^2(\theta)} + g\cos\theta + \text{constant} \tag{90}$$

I begin by seeking the stability of the 'sleeping' $\theta \approx 0$ state, note $p_\phi = p_\psi$ to avoid infinite PE at $\theta = 0$. Next: Taylor expand V:

$$V = \frac{p_\psi^2}{2I_1}\left(\frac{\theta^2}{4}\right) - g\left(\frac{\theta^2}{2}\right) + \text{constant} \tag{91}$$

$$= \frac{\theta^2}{2}\left(\frac{p_\psi^2}{4I_1} - g\right) \tag{92}$$

$$= \frac{\theta^2}{2}\left(\frac{I_3^2\omega_3^2}{4I_1} - g\right) \tag{93}$$

$$\tag{94}$$

For a stable equilibrium, the quantity in parenthesis must be positive. I'll wave my hands vigorously below to suggest that, to make a connection to the real rolling hurricane ball problem, one should actually use the original I_1&g, in which case the condition becomes:

$$\omega_3^2 > \frac{4I_1 g}{I_3^2} = 35g \tag{95}$$

which is way too big. (And note using $\tilde{I}_1$ and $\tilde{g}$ would have made things worse.)

Starting to wave my hands, I argue that there is a mechanism (rolling) that forces, as a holonomic constraint, $\psi = \phi$. The CM is a distance $a\cos\theta$ above the bottom ball's center which is a constant (a) above the plane, so PE$=mga\cos\theta$ works. (No need for $\tilde{g}$.) The situation actually keeps the CM from moving much, and in particular not rotating horizontally at $\dot{\phi}$ in a circle of radius $2a\sin\theta$ (which is what made $\tilde{I}_1$). If CM can be thought of as attached to a fixed vertical wire so it can only move vertically and $\dot{z} = -a \sin\theta\dot{\theta}$ whose KE contribution should be $\frac{1}{2}2ma^2 \sin^2\theta\,\dot{\theta}^2$, giving us an 'effective Lagrangian'

$$L = \frac{1}{2}\, I_1 \left(\dot{\phi}^2 \sin^2(\theta) + \dot{\theta}^2\right) + \frac{1}{2}\, I_3\, \dot{\phi}^2 \left(\cos(\theta) + 1\right)^2 + \frac{1}{2} \sin^2\theta\,\dot{\theta}^2 - g\cos\theta \tag{96}$$

Note that ϕ is ignorable, so we have p_ϕ a constant proportional to $\dot{\phi}$ (but also depending on θ). Focusing on the θ equation, and seeking a solution $\theta =$constant, we have:

$$I_1\dot{\phi}^2 \sin\theta\cos\theta - I_3\dot{\phi}^2(\cos\theta + 1)\sin\theta + \dot{\theta}^2 \sin\theta\cos\theta + g\sin\theta = \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = 0 \tag{97}$$

$$\dot{\phi}^2 \left((I_1 - I_3)\cos\theta - I_3\right) + g = 0 \tag{98}$$

$$(I_1 - I_3)\cos\theta = I_3 - \frac{g}{\dot{\phi}^2} \tag{99}$$

$$\cos\theta = \frac{2}{5} - \frac{g}{\dot{\phi}^2} \tag{100}$$

So at least that is as desired, however the terms that would be on the rhs (related to $\dot{\theta}$ and $\ddot{\theta}$) do not match Eq. (81) so the resulting dynamics are wrong. In short, by waving our hands the Lagrangian can get one thing right, but most everything else predicted by the Lagrangian is wrong.

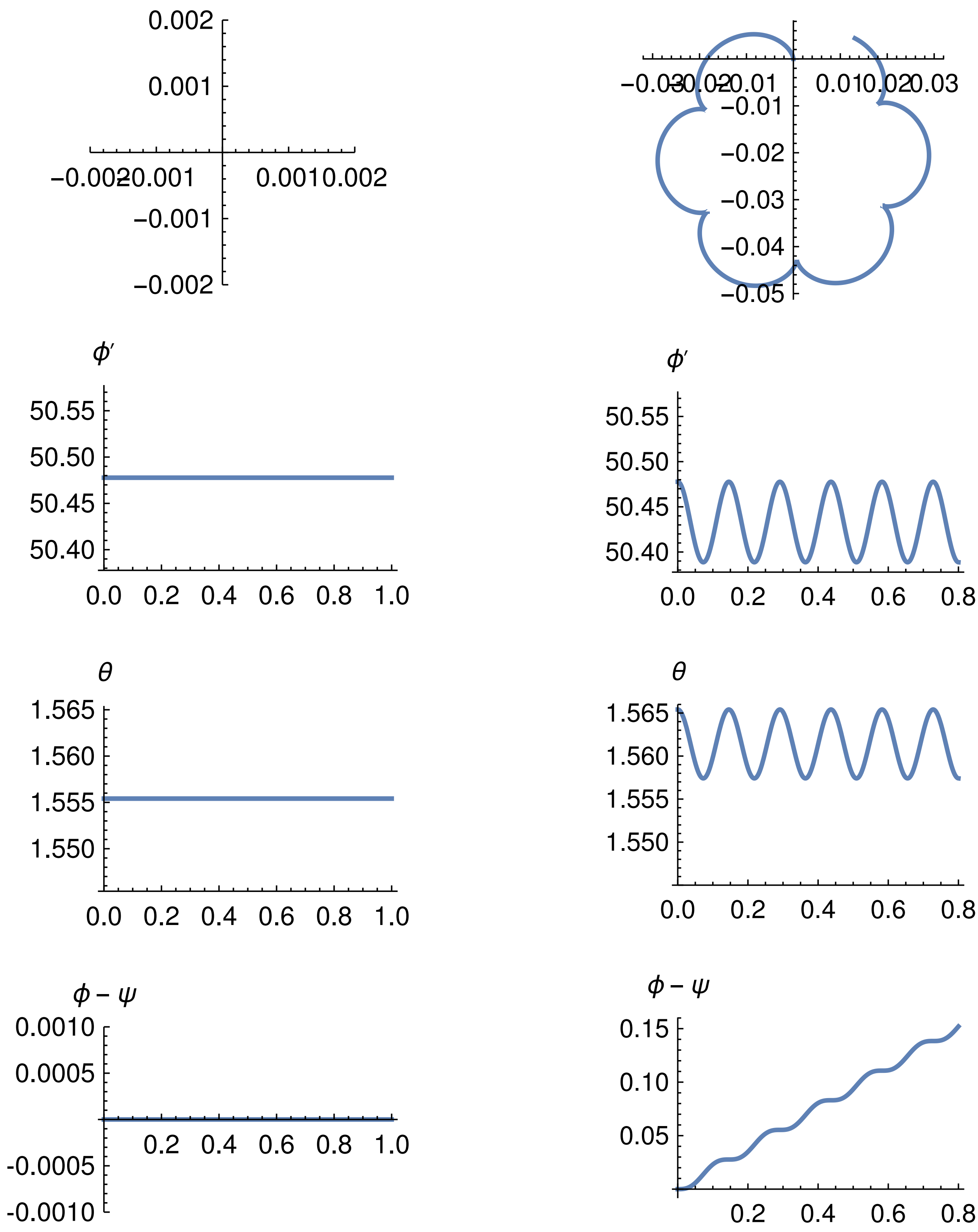

Figure 15: LHS: Starting with exact initial conditions for a θ =constant solution with $\dot{\psi}_0 = \dot{\phi}_0 \approx \sqrt{2.6g/a}$ (i.e., near the slow spin limit $\dot{\phi}_0 = \sqrt{2.5g/a}$) as expected all quantities are static with no CM motion
RHS: With θ slightly off we get oscillations. The solution is stable

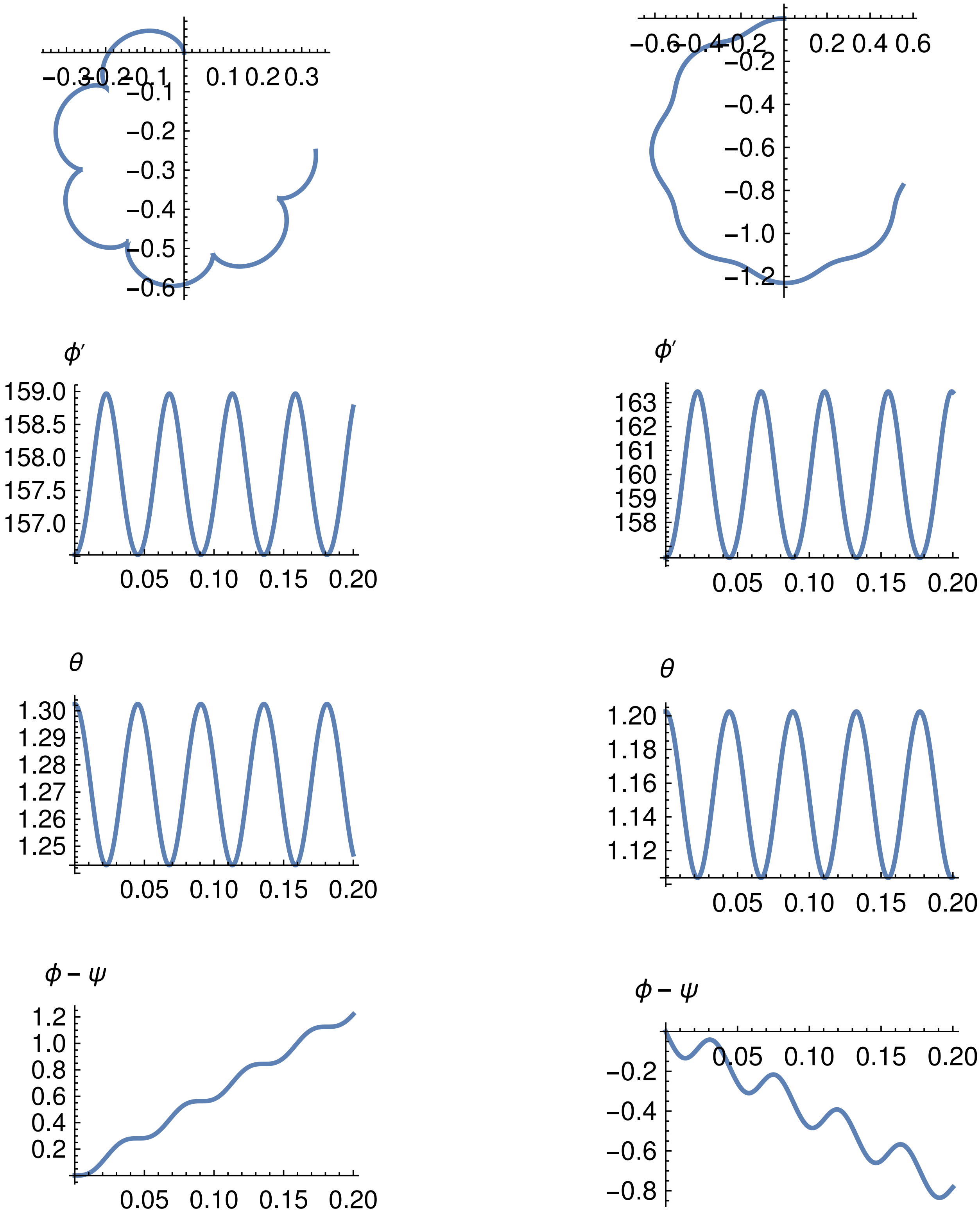

Figure 16: Same as previous but with much faster spin: $\dot{\psi}_0 = \sqrt{25g/a}$, and hence $\theta_0 = 1.20$ near the limit $\cos^1(\frac{2}{5}) = 1.16$

LHS: Starting with θ .01 above the exact initial conditions for a static solution

RHS: Starting with $\dot{\psi}_0$ 10% above that for a static solution. Note that θ bounces to angles smaller than the ‘limit’.

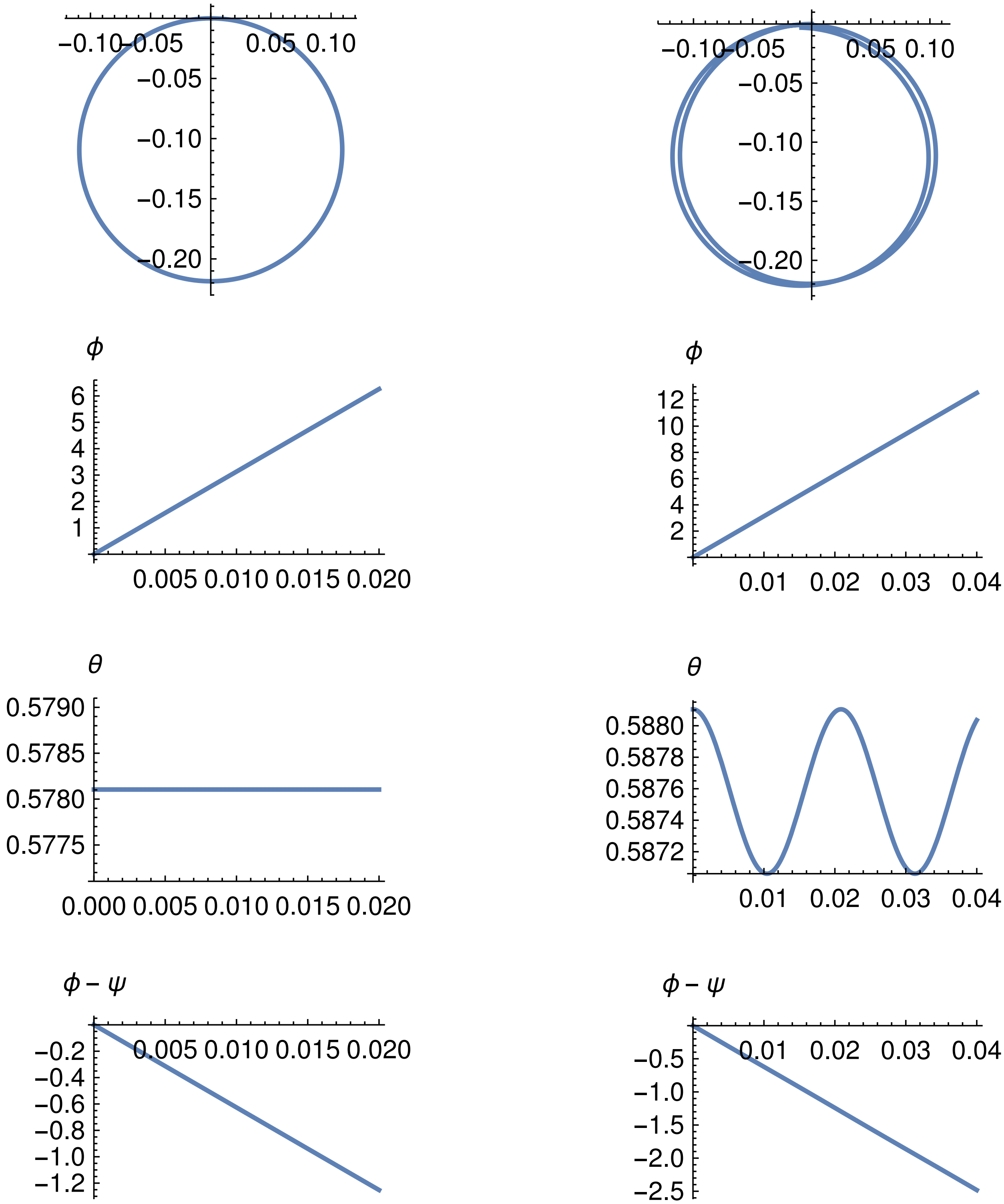

Figure 17: Solutions with smaller θ (but CM motion): $\rho = 1.2, \theta = 0.578$
LHS: Starting with the exact initial conditions for a static (circular) solution
RHS: Starting with θ 0.01 above that for a static solution.

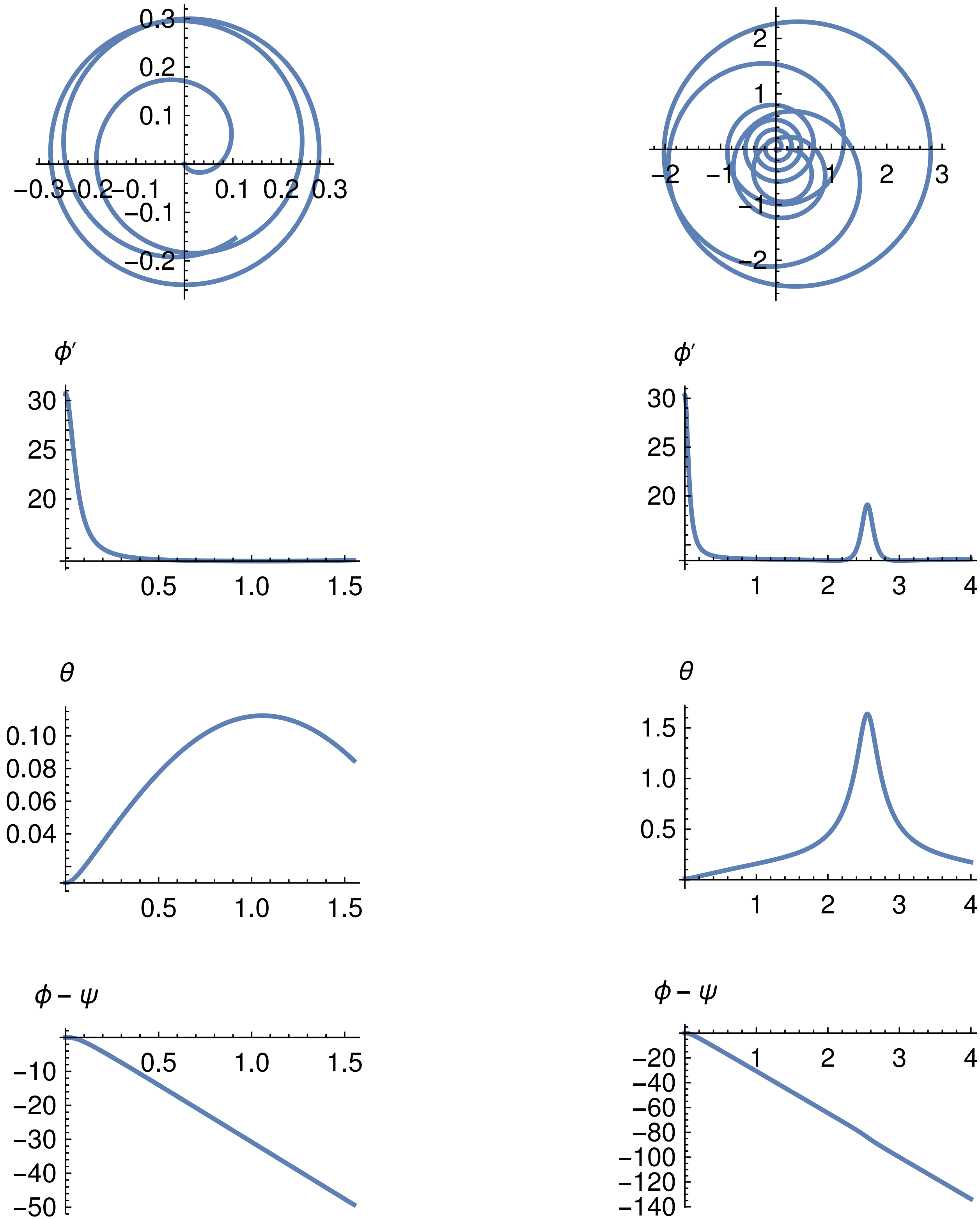

Figure 18: 'Sleeping spin' solutions, where the balls are started with near $\theta_0 = 0$, with $\dot\psi_0 = \dot\phi_0 \approx \sqrt{g/a}$

LHS: With $\dot\phi_0 = .98\sqrt{g/a}$, θ oscillates.

RHS: With $\dot\phi_0 = .97\sqrt{g/a}$, the CM spirals outward; θ eventually exceeds the limit of $\pi/2$.

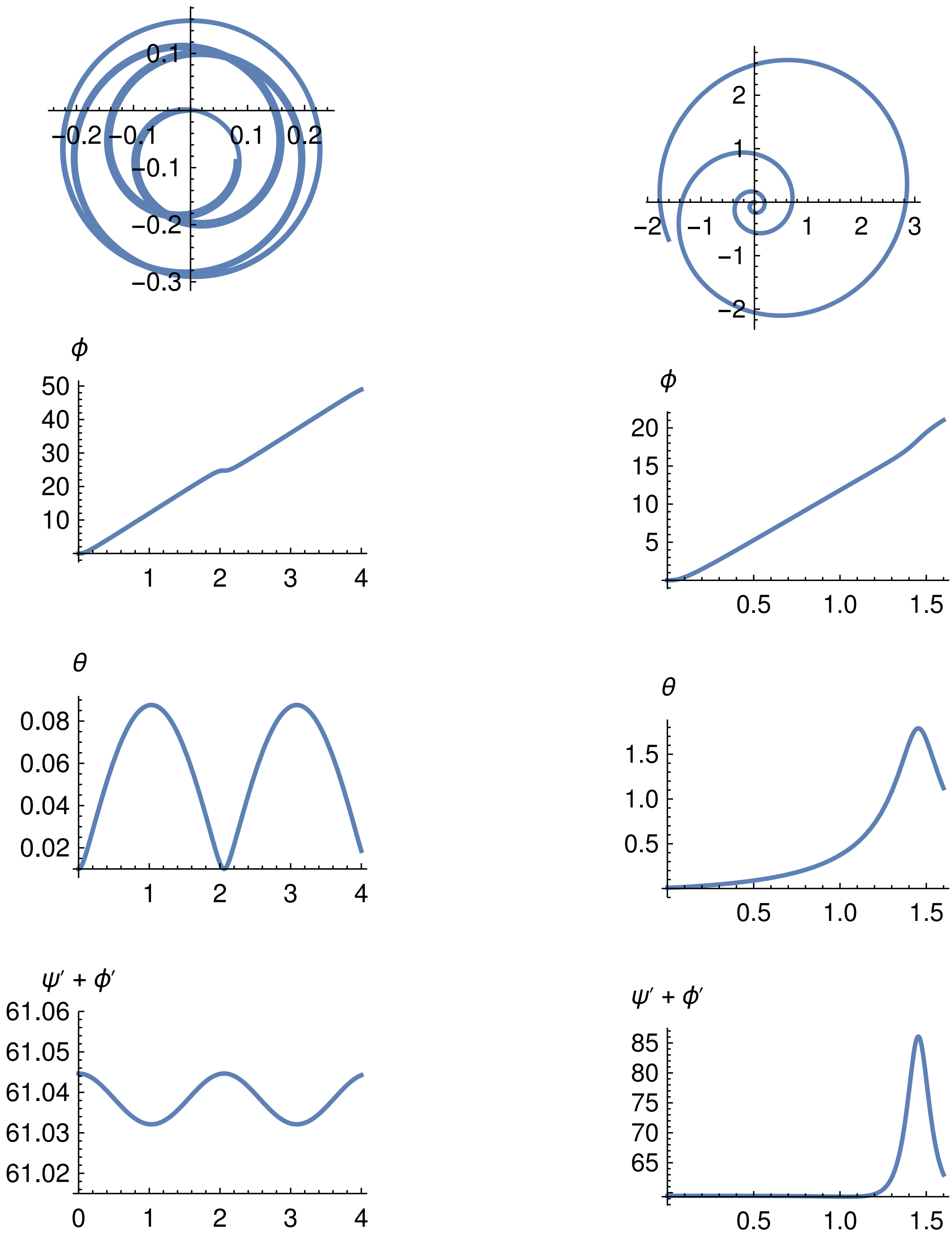

Figure 19: Same as previous except here initial conditions $\dot{\phi}_0 = 0$, $\dot{\psi}_0 \approx 2\sqrt{g/a}$. For $\theta \approx 0$, $(\phi + \psi)$ controls the configuration. From the third component of Eqs. (81–2), one can show $\dot{\psi} + \dot{\phi} \approx$constant. While $\dot{\psi}$ and $\dot{\phi}$ separately vary wildly the sum is nearly constant.
LHS: With $\dot{\psi}_0 = 1.95\sqrt{g/a}$, θ oscillates.
RHS: With $\dot{\psi}_0 = 1.9\sqrt{g/a}$, the CM spirals outward; θ eventually exceeds the limit of $\pi/2$.

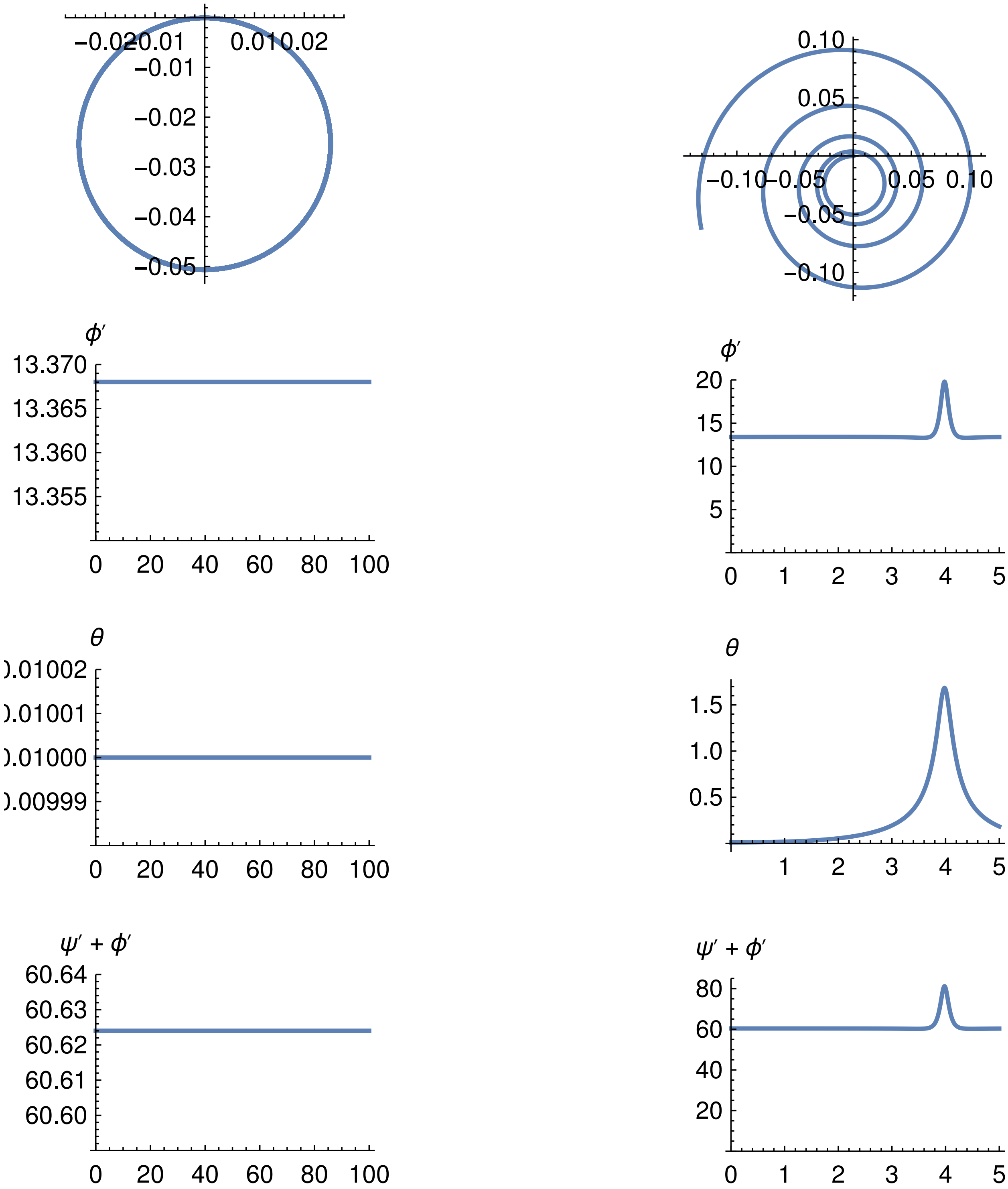

Figure 20: Seeking minimum $\dot{\psi}+\dot{\phi}$, 'sleeping' ($\theta \approx 0$) solutions (here near $\rho = 3.5$).
LHS: If Eq. (82) is exactly satisfied (here with $\theta_0 = .01, \rho = 3.535$) we find static solutions which are stable against small deviations.
RHS: With larger deviations the CM spirals outward; θ eventually exceeds the limit of $\pi/2$.