2 Particle Electrostatic PE the Hard Way

Since voltage is PE/charge, it is immediately clear that if a charge q is located where the voltage is ϕ the corresponding PE is $q\phi$. For charge distributed over a region, the PE is easily calculated as

$$U = \frac{1}{2}\int \rho(\mathbf{r})\phi(\mathbf{r})\;dV$$

(the $\frac{1}{2}$ coming from the assembly of the charge distribution from nothing) which using some clever vector calculus can be shown to be

$$U = \frac{1}{2}\int \mathbf{E}\cdot\mathbf{D}\;dV$$

The former form is almost always easier to calculate as the charge usually resides in a limited region of space whereas the electric field generally extends over all space. Additionally the idealization of 'point charges' produces infinities as $r \to 0$. Of course point charges don't exist, but the actual electrostatic self-energy of an electron is unknown. (A simple model of an electron as a uniformly charged sphere of radius R produces a self-energy of $\frac{3}{5}e^2/4\pi\epsilon_0 R$, which —while probably not far off the mark— is in detail certainly wrong. We do not currently know how much of the rest mass of an electron is due to its self electrostatic energy.)

Problem 6-7 provides an out. While the self-energy of charged particles is not known, it should be a constant and additive constants to PE do not affect forces/torques (since they are derivatives of PE). The plan is to separate the self energies (which are presumably actually finite but infinite in this 'point charge' model) and focus on the PE that depends on position (and which luckily remains finite in the 'point charge;' model). We consider two charges q_1 and q_2; q_1 is at the origin and q_2 is a distance d up the z axis from q_1.

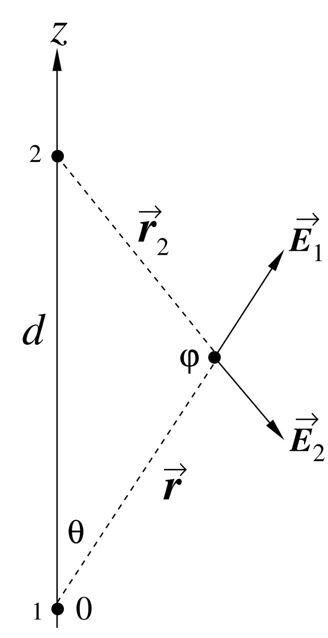
$$(\mathbf{E}_1 + \mathbf{E}_2)^2 = E_1^2 + E_2^2 + 2\mathbf{E}_1\cdot\mathbf{E}_2$$

The integrals of E_i^2 will be the respective (position independent) self energies, so we need only the $\mathbf{E_1}\cdot\mathbf{E_2}$ integral:

$$U_{12} = \epsilon_0\left(\frac{q_1 q_2}{(4\pi\epsilon_0)^2}\right)\int \frac{1}{r^2}\,\frac{1}{r_2^2}\;\cos\phi\;dV$$

The following relationships follow from the law of cosine for the triangle connecting the two charges and an arbitrary point in space ($\mathbf{r}$ from the origin)

$$\begin{aligned} r_2^2 &= r^2 + d^2 - 2rd\cos\theta \\ d^2 &= r^2 + r_2^2 - 2rr_2\cos\phi \\ \cos\phi &= \frac{r^2 + r_2^2 - d^2}{2rr_2} \end{aligned}$$



Note the $\cos\phi$ is also the needed quantity for the dot product $\mathbf{E_1}\cdot\mathbf{E_2}$. We use the common trick for azimuthally symmetric integrals in spherical coordinates:

$$dV = r^2 dr\;\sin\theta\,d\theta\;d\phi = -r^2 dr\;dc\;2\pi$$

where $c = \cos\theta$ whose range is $c \in (-1, 1)$.

In the following *Mathematica* code: `r2`$=r_2^2$, `c`$=\cos\theta$

```
r2=r^2+d^2-2 r d c
c2=(r^2+r2-d^2)/(2 r Sqrt[r2])
Integrate[1/r2 c2 2 Pi ,{c,-1,1}]
Simplify[%,Assumptions->r<d&&r>0]
0
Simplify[%%,Assumptions->d<r&&d>0]

         4 Pi
Out[5]=  ----
           2
          r

Integrate[% ,{r,d,Infinity}]

                                4 Pi
Out[6]=  ConditionalExpression[----, Im[d] != 0 || Re[d] > 0]
                                 d
```

From which we conclude:

$$
\begin{aligned}
U_{12} &= \epsilon_0 \left(\frac{q_1 q_2}{(4\pi\epsilon_0)^2} \right) \frac{4\pi}{d} \\
&= \frac{q_1 q_2}{4\pi\epsilon_0 d}
\end{aligned}
$$

exactly as we could have immediately written down from PE.