Multipole Expansion Example 2

Consider a not-too-simple simple charge distribution: a uniformly charged rod of length $2a$ that sits on the z axis with center at the origin. Since we're going to be doing a good bit of algebra, simple expressions are helpful. In this particular problem everywhere there would be an overall factor of

$$\frac{\lambda}{4\pi\epsilon_0}$$

where λ is the charge per length of the rod. We will ignore this overall factor. Without loss of generality we seek the voltage at the 'arbitrary' point $\mathbf{r} = (x, 0, z)$ by integrating over the source (the charged rod) $\mathbf{r}' = (0, 0, z')$:

$$\phi = \int_{-a}^{+a} \frac{dz'}{\sqrt{x^2 + (z - z')^2}}$$

We will eventually do this integral, but if we couldn't we could always approximate the field using the multipole expansion.

The monopole term (total charge) is easy: $\lambda 2a$; having pulled out the above overall factor that term gives:

$$\phi = \frac{2a}{r}$$

There is no dipole term as the center of charge is the origin.

So the fun starts with the quadrupole term... we just need to integrate over the source distribution:

$$Q_{ij} = \int \left(3x'_i x'_j - r'^2 \delta_{ij}\right) \lambda \, dz'$$

where $\mathbf{r}' = (0, 0, z')$.

```
Q=3 Outer[Times,{x,y,z},{x,y,z}]-(x^2+y^2+z^2) IdentityMatrix[3]
Q /. {x->0,y->0}
Integrate[% ,{z,-a,a}]

                -2              -2                4
Out[3]= a^3 {{---, 0, 0}, {0, ---, 0}, {0, 0, ---}}
                3               3                 3
```

$$Q = \frac{2}{3} a^3 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Since our axes are aligned with the symmetry of the object, Q is diagonal, and of course $\mathrm{Tr}(Q) = 0$.

The multipole expansion then says the resulting term in the electric potential is

$$\phi = \frac{\hat{\mathbf{r}} \cdot Q \cdot \hat{\mathbf{r}}}{2r^3}$$

For our field observation point $\hat{\mathbf{r}} = (\sin\theta, 0, \cos\theta)$:

```
{Sin[t],0,Cos[t]}.%3.{Sin[t],0,Cos[t]}/(2 r^3)
Simplify[%]

          3
         a  (1 + 3 Cos[2 t])
Out[5] = -------------------
                    3
                 6 r
phi2=% + 2 a/r
```

This is a simple enough charge distribution that *Mathematica* can do the general integral (which allows us to compare to the approximation):

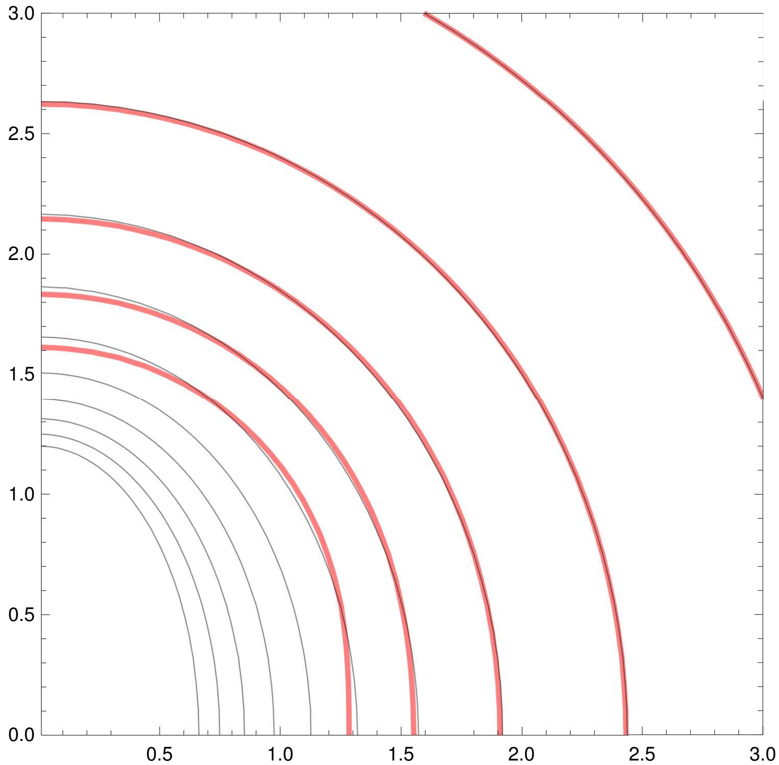
```
Integrate[1/Sqrt[x^2+(z-z1)^2],{z1,-a,a},Assumptions-> x>0&&z>0&&a>0]

phi=Simplify[% /. {z-> r Cos[t],x->r Sin[t]} ]
Series[phi,{r,Infinity,5}]
Simplify[%]

                 3                      5
         2 a    a  (1 + 3 Cos[2 t])    a  (9 + 20 Cos[2 t] + 35 Cos[4 t])       -6
Out[11]= --- + ------------------- + ---------------------------------- + O[r]
          r              3                                   5
                      6 r                               160 r
```

The first two terms match the monopole and quadrupole terms we found from the charge distribution, the third would be the 16-pole term.

We can compare the approximate voltage (red) to the exact result:



It seems for $r > 2a$ the approximation closely follows the exact result.

Notice that the sign of the rod's quadrupole is the opposite of the ring's. If desired we could design a superposition problem where the quadrupoles cancelled making a charge distribution whose field would be monopole plus 16-pole (and so nearly pure monopole). Or we could use opposite-sign charges on rod and ring to kill the monopole term and make a charge distribution that started quadrupole.