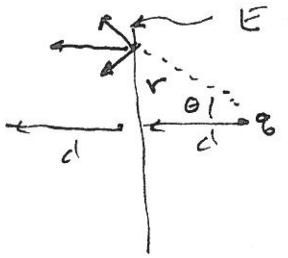
341 – Class 12 – ch 3   3-14, 20 relax

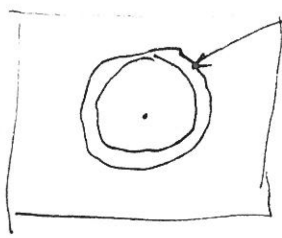
14) 

$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{d^2+r^2}$ ; normal component $E_{total} = 2E\cos\theta$

$$= \frac{2q}{4\pi\varepsilon_0}\frac{\cos\theta}{d^2+r^2} \qquad \frac{d}{\sqrt{d^2+r^2}}$$

$$\sigma = \varepsilon_0 E_n = \frac{q}{2\pi}\frac{d}{(d^2+r^2)^{3/2}}$$

face on



annulus: area $= 2\pi r\,dr$ ; $dq = \sigma\, 2\pi r\,dr$

$$dq_i = qd\frac{r\,dr}{(d^2+r^2)^{3/2}} \qquad u = r^2$$

$$q_i = \frac{qd}{2}\int_0^\omega \frac{du}{(d^2+u)^{3/2}} = \frac{qd}{2}\left.\frac{(d^2+u)^{-1/2}}{-1/2}\right|_0^\omega$$

$$= qd\left(\frac{1}{(d^2)^{-1/2}} - \frac{1}{\omega}\right) = q \checkmark$$

mathematica code for derivative in #20

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```
In[1]:= f[x_]=q1/Sqrt[r^2+x^2-2 x r Cos[t]]+q2/Sqrt[d^2+x^2-2 x d Cos[t]]

                    q2                                   q1
Out[1]= ---------------------------- + ---------------------------
             2    2                          2    2
        Sqrt[d  + x  - 2 d x Cos[t]]   Sqrt[r  + x  - 2 r x Cos[t]]

In[2]:= D[f[x],x]

          -(q2 (2 x - 2 d Cos[t]))            q1 (2 x - 2 r Cos[t])
Out[2]= ------------------------------ - ----------------------------
            2    2                 3/2        2    2                 3/2
        2 (d  + x  - 2 d x Cos[t])        2 (r  + x  - 2 r x Cos[t])

In[2]:= % /. {x->R, d->R^2/r, q2->-q1 R/r}
                                                        2
In[4]:= Simplify[%, Assumptions->r<R&&0<r]

                   2    2
              q1 (r  - R )
Out[4]= ----------------------------
            2    2                3/2
        R (r  + R  - 2 r R Cos[t])
```

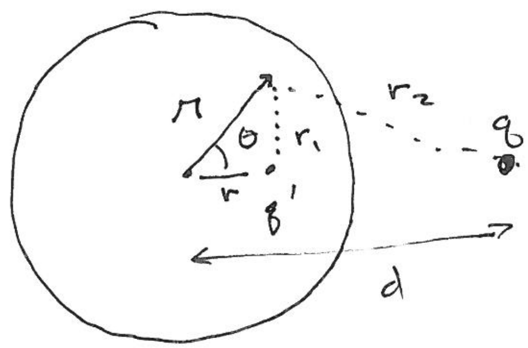
20) we previously were working $q$ outside of sphere — found imagined $q'$ such that $V=0$ on surface — Now we are interested in inside: $q'$ becomes real & $q$ imagined but same algebra shows $V=0$ ...

Before: $d' = \frac{R^2}{d}$ Now $d' = r$ & $d = \frac{R^2}{r}$

$q' = -\frac{R}{d} q$ $q = -\frac{d}{R} q' = -\frac{R}{r} q'$

notation: $r$ — location of real charge $q'$
$d$ — location of image charge $q$
$\mathcal{r}$ — any real distance from center of sphere where we seek voltage ($\mathcal{r} < R$ for real)
$R$ — radius of actual conducting sphere

Review of proof $V=0$ with the new point-of-view




distance twixt $q'$ & observation pt

$r_1^2 = r^2 + \mathcal{r}^2 - 2\mathcal{r} r \cos\theta$

$r_2^2 = d^2 + \mathcal{r}^2 - 2\mathcal{r} d \cos\theta$

$$4\pi\varepsilon_0 \phi = \frac{q'}{r_1} + \frac{q}{r_2}$$

to show is zero on sphere ie $\mathcal{r} = R$

$$4\pi\varepsilon_0 \phi\Big|_{\mathcal{r}=R} = \frac{q'}{\sqrt{r^2 + R^2 - 2Rr\cos\theta}} + \frac{q}{\sqrt{d^2 + R^2 - 2Rd\cos\theta}}$$

$$= \frac{q'/R}{\sqrt{\left(\frac{r}{R}\right)^2 + 1 - 2\frac{r}{R}\cos\theta}} + \frac{q/d}{\sqrt{1 + \left(\frac{R}{d}\right)^2 - 2\left(\frac{d}{R}\right)\cos\theta}}$$

Now if $\frac{r}{R} = \frac{R}{d}$ denom match for all $\theta$

so numer match/cancel

make $\frac{q'}{R} = -\frac{q}{d}$

$$q = -\frac{d}{R} q' = -\frac{R}{r} q'$$

$d = \frac{R^2}{r}$

to calculate $\sigma$ need $E_r\Big|_{\mathcal{r}=R} = -\partial_{\mathcal{r}}\phi$ note the normal to conducting sphere points inward (ie $\hat{n} = -\hat{r}$)

so $E_r > 0$ means $\sigma < 0$ ... $\sigma = +\varepsilon_0 \partial_{\mathcal{r}} \phi\Big|_R$

$$4\pi\varepsilon_0 \partial_r \phi \Big|_{r=R} = \frac{q'(r^2-R^2)}{R\left[r^2+R^2-2rR\cos\theta\right]^{3/2}}$$ mathematica

$$\sigma = \frac{1}{4\pi} q' \frac{(r^2-R^2)}{R} \frac{1}{\left[r^2+R^2-2rR\cos\theta\right]^{3/2}}$$

surface area element of sphere $= R^2 \sin\theta\, d\theta\, 2\pi$

total induced charge on inner surface of sphere $Q$

$$Q = \frac{1}{2} q'(r^2-R^2) R \int \frac{\sin\theta\, d\theta}{\left[r^2+R^2-2rR\cos\theta\right]^{3/2}}$$ $c = \cos\theta$

$$= \frac{1}{2} q'(r^2-R^2)R \int_{-1}^{1} \frac{dc}{\left[r^2+R^2-2rRc\right]^{3/2}}$$

$$= \frac{1}{2} q'(r^2-R^2)R \left[ \frac{\left[r^2+R^2-2rRc\right]^{-1/2}}{rR} \right]_{-1}^{1}$$

note $r < R$

$$= \frac{1}{2} q' \frac{(r^2-R^2)}{r} \left[ \frac{1}{R-r} - \frac{1}{R+r} \right]$$

$$\hookrightarrow \frac{(R+r)-(R-r)}{(R^2-r^2)} = \frac{2r}{(R^2-r^2)}$$

$$= -q'$$

this had to be true via gaussian surface embedded in conducting sphere
$E=0 \Rightarrow$ total enclosed charge $=0$
$\Rightarrow$ induced charge must match real $q'$