Addition of Angular Momentum (aka Clebsch-Gordan vector coupling)

Remark: Recall from classical mechanics the connection between symmetry & conservation. If a coordinate q does not appear in Lagrangian ($\frac{\partial L}{\partial q} = 0$) then the corresponding momentum is constant ($\frac{\partial L}{\partial \dot{q}} = \text{const}$). In QM symmetry corresponds to $[H, Q] = 0 \Rightarrow$ degeneracy likely as ψ eigenfunction $\Rightarrow$ $Q\psi$ eigenfunction with same energy. Recall also the important relation between $\vec{L}$ & rotation — that $\vec{L}$ "generates" rotations of the $\vec{r}$ & $\vec{p}$ that make it up. $[H, \vec{L}] = 0$ was start of program to show the "m degeneracy" — $2l+1$ states with different orientation but the same L^2

Note: the only thing we used in producing the $|lm\rangle$ states and $L_\pm$ operators was the commutation relation

(A) $[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$. If we discover new operators that follow that relations all the rest follows: $2l+1$ eigenfunctions with L^2 eigenvalue $\hbar^2 l(l+1)$ & L_z eigenvalue $m\hbar$

Consider orbital angular momentum operator $\vec{L}$ & spin angular momentum operator $\vec{S}$. They operate on separate coordinates (CM location or Euler angle orientation) and so commute with each other. Separately they satisfy (A) so the sum $\vec{J} = \vec{L} + \vec{S}$ also satisfies (A) and hence there must be a sequence of eigenfunctions of J^2 with eigenvalue we call j: $\hbar^2 j(j+1)$ & J_z eigenvalue $\hbar m_j$. For reasons I hope will be explained below — we want to find these eigenfunctions.

Consider two particles with coordinates $\vec{r}_1$ & $\vec{r}_2$. There will be corresponding angular momentum operators $\vec{L}_1$ & $\vec{L}_2$. Since they operate on different coordinates they will commute; separately they satisfy (A), so the sum $\vec{L} = \vec{L}_1 + \vec{L}_2$ also satisfies (A) and so there must exist a set of states with the same eigenvalue of L^2 and different (stepwise) eigenvalues of L_z

The previous examples aim to show that given two angular momentum operators we can form a sum angular momentum which must satisfy the rules/states generated by the raising/lowering operators. But why bother?
In the context of two orbiting particles (say in He atom) rotating the coordinates of just one particle will generally not produce an equivalent problem — ie $[H, \vec{L}_1] \neq 0$ but rotating the coordinates of every particle will leave an equivalent problem — ie $[H, \vec{L}_1 + \vec{L}_2] = 0$ and in that case $(L_1 + L_2)_\pm$ will generate degenerate states whereas $L_{1\pm}$ will not.
Remark: if we ignore the electrostatic interaction between the two electrons in He atom $\left(\frac{e^2}{4\pi\varepsilon_0 |\vec{r}_1 - \vec{r}_2|}\right)$ then in fact $[H, \vec{L}_1] = 0$
In the case of a single orbiting & spinning electron, if we rotate just $\vec{L}$ (and leave $\vec{S}$) the angle between $\vec{L}$ & $\vec{S}$ will change & it turns out the energy H does depend on the relative orientation of $\vec{L}$ & $\vec{S}$: $\vec{L} \cdot \vec{S}$ whereas if we rotate $\vec{L}$ & $\vec{S}$ by the same angle $\vec{L} \cdot \vec{S}$ is invariant. This is a long way of saying $[H, \vec{L}] \neq 0$ & $[H, \vec{S}] \neq 0$ but $[H, \overrightarrow{L+S}] = 0$, so the eigenfunctions of $\overrightarrow{L+S}$ will form a degenerate set.
In the case of a pair of isolated, spinning electrons we expect an interactions (ie, one electron makes a $\vec{B}$ the other then has $PE = -\vec{\mu} \cdot \vec{B}$) But clearly that interaction depends only on the relative orientation of the spins ie $\vec{S}_1 \cdot \vec{S}_2$. Our dot product commutation rule then shows $[\vec{S}_1 \cdot \vec{S}_2, \overrightarrow{S_1 + S_2}] = 0$ so eigenfunctions of total spin will be degenerate.
Counter example: spinning electrons in an <u>external</u> $\vec{B}$: rotating $\vec{S}$ (but not $\vec{B}$) changes energy & $[H, \vec{S}] \neq 0$ degeneracy rules do not apply

Example: pair of isolated spin electrons with energy $\propto \vec{S}_1 \cdot \vec{S}_2$

Electrons are spin $\frac{1}{2}$ so $l = s = \frac{1}{2}$ & $m_s = +\frac{1}{2}$ or $-\frac{1}{2}$

notation $|s\ m_s\rangle = |\tfrac{1}{2}\ \tfrac{1}{2}\rangle = \uparrow = \chi_+$ text

$|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = \downarrow = \chi_-$ text

We have product wavefunctions where first arrow is always (1)

Trick: $\vec{S}_1 \cdot \vec{S}_2 = \underbrace{S_{1x} S_{2x} + S_{1y} S_{2y}}_{\text{compare to}} + S_{1z} S_{2z}$

$$S_{1+} S_{2-} + S_{1-} S_{2+} = (S_{1x} + iS_{1y})(S_{2x} - iS_{2y}) + (S_{1x} - iS_{1y})(S_{2x} + iS_{2y})$$

$$= S_{1x} S_{2x} + S_{1y} S_{2y} + i(S_{1y} S_{2x} - S_{1x} S_{2y})$$

$$+ S_{1x} S_{2x} + S_{1y} S_{2y} - i(S_{1y} S_{2x} - S_{1x} S_{2y})$$

$$= 2(S_{1x} S_{2x} + S_{1y} S_{2y})$$

so $\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}(S_{1+} S_{2-} + S_{1-} S_{2+}) + S_{1z} S_{2z}$

general case: $L_\pm |l\ m\rangle = \hbar \sqrt{l(l+1) - m(m \pm 1)}\ |l\ m \pm 1\rangle$

$S_+ \uparrow = 0$

$S_+ \downarrow = \hbar \sqrt{\frac{3}{4} - -\frac{1}{2}(\frac{1}{2})}\ \uparrow = \hbar \uparrow$

$S_- \uparrow = \hbar \sqrt{\frac{3}{4} - \frac{1}{2}(-\frac{1}{2})}\ \downarrow = \hbar \downarrow$

$S_- \downarrow = 0$

we have 4 product states: $\uparrow\uparrow,\ \uparrow\downarrow,\ \downarrow\uparrow,\ \downarrow\downarrow$

Seek how $\vec{S}_1 \cdot \vec{S}_2$ affects each state

$$\left[\tfrac{1}{2}(S_{1+} S_{2-} + S_{1-} S_{2+}) + S_{1z} S_{2z}\right] \uparrow\uparrow = \hbar^2 \tfrac{1}{4} \uparrow\uparrow$$ ← eigen function

$$\left[\tfrac{1}{2}(S_{1+} S_{2-} + S_{1-} S_{2+}) + S_{1z} S_{2z}\right] \downarrow\downarrow = \hbar^2 \tfrac{1}{4} \downarrow\downarrow$$ ← degenerate eigen function

$$\left[\tfrac{1}{2}(S_{1+} S_{2-} + S_{1-} S_{2+}) + S_{1z} S_{2z}\right] \uparrow\downarrow = \hbar^2 \left\{\tfrac{1}{2} \downarrow\uparrow - \tfrac{1}{4} \uparrow\downarrow\right\}$$

$$\left[\tfrac{1}{2}(S_{1+} S_{2-} + S_{1-} S_{2+}) + S_{1z} S_{2z}\right] \downarrow\uparrow = \hbar^2 \left\{-\tfrac{1}{4} \downarrow\uparrow + \tfrac{1}{2} \uparrow\downarrow\right\}$$

see that $[\ ](\uparrow\downarrow + \downarrow\uparrow) = \hbar^2 \left\{\tfrac{1}{4} \downarrow\uparrow + \tfrac{1}{4} \uparrow\downarrow\right\}$ ← degenerate eigenfunction

see that $[\ ](\uparrow\downarrow - \downarrow\uparrow) = \hbar^2 \left\{-\tfrac{3}{4} \uparrow\downarrow + \tfrac{3}{4} \downarrow\uparrow\right\}$

eigenvalue $= -\frac{3}{4}\hbar^2$

Faster: angular momentum addition: $\overrightarrow{\frac{1}{2}+\frac{1}{2}} = \underbrace{0,1}$ "S" 3x degenerate

$$\vec{S}_1\cdot\vec{S}_2 = \frac{1}{2}\left[(\overrightarrow{S_1+S_2})^2 - S_1^2 - S_2^2\right]$$
$$= \frac{\hbar^2}{2}\left[s(s+1) - \frac{3}{4} - \frac{3}{4}\right] \begin{array}{l} \xrightarrow{s=1} +\frac{\hbar^2}{4} \\ \xrightarrow{s=0} -\frac{3}{4}\hbar^2 \end{array}$$

Generalize: Given eigenstates of $\vec{L}$ $|lm\rangle$ for given l

Given eigenstates of $\vec{S}$ $|s m_s\rangle$ for given s

Construct product states. $|lm\rangle|s m_s\rangle$ and find combination of such states that is eigenfunction $\vec{J}=\vec{L}+\vec{S}$

Note: since $J_z|lm\rangle|sm_s\rangle = (L_z+S_z)|lm\rangle|sm_s\rangle$
$$= \hbar\,(\underbrace{m+m_s}_{m_j})|lm\rangle|sm_s\rangle$$

we need only consider product state combinations that have $m+m_s=$ fixed

Classically we would expect total angular momentum to range from $\overset{S}{\nearrow}$ $l+s$ to $\swarrow_S$ $|l-s|$ – true in QM

Trick: $J^2 = L^2+S^2+2\vec{L}\cdot\vec{S} = \underbrace{L^2}_{\hbar^2 l(l+1)}+\underbrace{S^2}_{\hbar^2 s(s+1)}+2\underbrace{L_zS_z}_{m m_s \hbar^2}+L_+S_-+L_-S_+$

Note: If we "start at the top" $m=l$ $m_s=s$ discover its already an eigenfunction of J^2

$$[L^2+S^2+2L_zS_z+\underbrace{L_+S_-}_{0}+\underbrace{L_-S_+}_{0}]|ll\rangle|ss\rangle = \hbar^2\left\{l(l+1)+s(s+1)+2ls\right\}|ll\rangle|ss\rangle$$

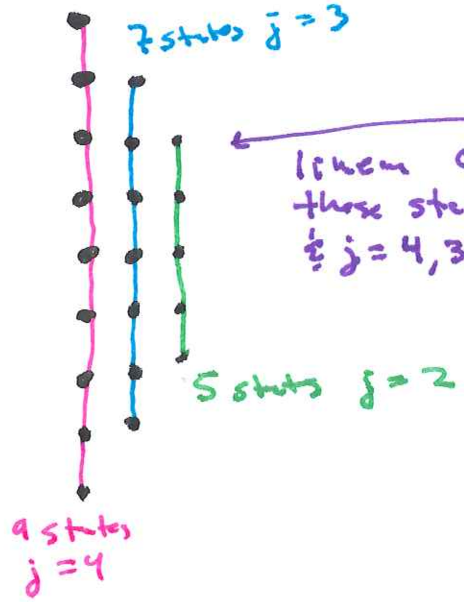
$l(l+s+1)+s(l+s+1)$

we can then lower this state $=(l+s)(l+s+1)$ with (L_-+S_-) to get the linear ie $j=l+s$ combo $|l\,l-1\rangle|ss\rangle$ & $|ll\rangle|s\,s-1\rangle$ that has $m_j=j-1$

the orthogonal combination of these two must have $j=l+s-1$, $m_j=l+s-1$

etc.

Example: $\ell=3$ $s=1$. We label states just with $m_\ell\, m_s$. Make rows of states with same $m_j = m_\ell + m_s$



linear combos of these states $\Rightarrow m_j = 2$ & $j = 4, 3, 2$

(3,1)
(2,1) (3,0)
(1,1) (2,0) (3,−1)
(0,1) (1,0) (2,−1)
(−1,1) (0,0) (1,−1)
(−2,1) (−1,0) (0,−1)
(−3,1) (−2,0) (−1,−1)
(−3,0) (−2,−1)
(−3,−1)

Note: the dimension of a vector space does not depend on basis: $(2\ell+1)(2s+1)$ product states so must have same number of $|j\, m_j\rangle$ states:

$$\sum_{j=\ell-s}^{\ell+s} 2j+1 = 2\left(\frac{2s+1}{2}\right)(2\ell) + (2s+1) = (2s+1)(2\ell+1) \checkmark$$

← Gauss Trick

Notation:

$$\underbrace{|\ell s j m_j\rangle}_{\substack{\text{state with total angular momentum}\\ j \text{ \& z component } m_j \\ \text{made up of states with } L\to\ell,\ S\to s}} = \sum_{\substack{m+m_s=m_j\\ \text{fixed}}} \overbrace{|\ell m\rangle |s m_s\rangle}^{\text{product state}} \overbrace{\langle \ell s\, m m_s | \ell s\, j m\rangle}^{\text{Clebsch-Gordan}}$$

Restatement: Only angular momentum that is conserved will generate degeneracy. Often only total momentum is conserved.