Note: For vectors it is useful to invent a notation that distinguishes unit vectors ($\hat{\mathbf{e}}$) from generic vectors ($\vec{\mathbf{v}}$). For those same reasons, below I've used $u(x)$ to denote a normalized wavefunction, i.e., $\int u^*(x)u(x)\,dx = 1$; just as a unit vector is a vector, $u(x)$ is a ψ. While this notation is not standard, I find it useful.

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2-\beta x}\,dx = \sqrt{\frac{\pi}{\alpha}}\,e^{\beta^2/4\alpha} \qquad \int_0^{+\infty} x\,e^{-\alpha x^2}\,dx = \frac{1}{2\alpha} \qquad \int_0^{\infty} x^n\,e^{-\alpha x}\,dx = n!/\alpha^{n+1}$$

$$H\psi = i\hbar\partial_t\psi \qquad H\psi = E\psi \qquad H = \frac{p^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\,\partial_x^2 + V(x) \qquad p = -i\hbar\partial_x \qquad [p,x] = -i\hbar$$

Notation: $Hu_n = E_n\,u_n$, $u_n(x)$ orthonormal, and $\omega_n \equiv E_n/\hbar:\ \Psi(x,t=0) = \sum c_n u_n(x) \Rightarrow \Psi(x,t) = \sum c_n u_n e^{-i\omega_n t}$

where (Fourier's Trick): $c_n = \langle u_n|\Psi\rangle$ Notation: $\langle f|Q|g\rangle \equiv \int_{-\infty}^{+\infty} f^*(x)\,Qg(x)\,dx$

$$\partial_t\,\psi^*(x,t)\psi(x,t) = -\partial_x J \qquad \text{where current } J = \frac{\hbar}{2im}\left(\psi^*\,\partial_x\psi - \psi\,\partial_x\psi^*\right) = \frac{\hbar}{2im}\,\psi^*\,\overleftrightarrow{\partial_x}\,\psi$$

$$\frac{d}{dt}\langle\psi|A|\psi\rangle = \langle\psi|\partial_t A|\psi\rangle + \langle\psi|i[H,A]/\hbar|\psi\rangle \qquad [A,B] \equiv AB - BA$$

$$\sigma_A\sigma_B = \Delta A\,\Delta B \geq \frac{1}{2}\left|\langle\psi|i[A,B]|\psi\rangle\right| \qquad [AB,C] = A[B,C] + [A,C]B$$

Free Particle: $\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\,e^{ipx/\hbar}$ or $\psi_k(x) = \frac{1}{\sqrt{2\pi}}\,e^{ikx}$ where $p = \hbar k$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ikx}\,g(k)\,dk \qquad g(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ikx}\,f(x)\,dx$$

Particle-in-a-box with $V(x) = 0$ for $|x| < a$, but $V(x) = \infty$ elsewhere. Note: $L = 2a$ $\quad n = 1,2,3\ldots$

$$E_n^+ = \frac{(k\hbar)^2}{2m} \qquad u_n^+(x) = \frac{1}{\sqrt{a}}\cos(kx) \qquad \text{where} \qquad k = \frac{(n-\frac{1}{2})\pi}{a} = \frac{(2n-1)\pi}{2a} = \frac{(\text{odd }\mathbb{N})\pi}{2a}$$

$$E_n^- = \frac{(k\hbar)^2}{2m} \qquad u_n^-(x) = \frac{1}{\sqrt{a}}\sin(kx) \qquad \text{where} \qquad k = \frac{n\pi}{a} = \frac{(2n)\pi}{2a} = \frac{(\text{even }\mathbb{N})\pi}{2a}$$

shifted origin: $E_n = \frac{(k\hbar)^2}{2m}$ $\quad u_n(x) = \sqrt{\frac{2}{L}}\,\sin(kx)$ $\quad$ where $\quad k = \frac{n\pi}{L}$

Particle-in-a-box with $V(x) = -V_0$ for $|x| < a$, but $V(x) = 0$ elsewhere

$$z_0^2\,\frac{(\hbar/a)^2}{2m} = V_0 \qquad -\frac{\kappa^2\hbar^2}{2m} = E \qquad z^2\,\frac{(\hbar/a)^2}{2m} = \frac{k^2\hbar^2}{2m} = E + V_0 = KE \qquad \text{where: } z = ka$$

$$E_n^+ = \left(z^2 - z_0^2\right)\frac{(\hbar/a)^2}{2m} \qquad \psi_n^+(x) = \begin{cases} \cos(kx) & |x| < a \\ e^{-\kappa|x|} & |x| > a \end{cases} \qquad \text{where} \qquad \tan z = \sqrt{(z_0/z)^2 - 1} \qquad z \approx \frac{(\text{odd }\mathbb{N})\pi}{2}$$

$$E_n^- = \left(z^2 - z_0^2\right)\frac{(\hbar/a)^2}{2m} \qquad \psi_n^-(x) = \begin{cases} \sin(kx) & |x| < a \\ \pm e^{-\kappa|x|} & |x| > a \end{cases} \qquad \text{where} \qquad \cot z = -\sqrt{(z_0/z)^2 - 1} \qquad z \approx \frac{(\text{even }\mathbb{N})\pi}{2}$$

$$\text{reflection: (shifted origin)} \quad \psi(x) = \begin{cases} e^{ikx} + Re^{-ikx} & x < 0 \\ A\cos qx + B\sin qx & 0 < x < L \\ Te^{ikx} & x > L \end{cases} \qquad \text{where} \qquad R = \frac{i(q^2-k^2)\sin qL}{2qk\cos qL - i(q^2+k^2)\sin qL}$$

Harmonic Oscillator with $V(x) = \frac{1}{2}\,m\omega^2 x^2$ $\quad E_n = \hbar\omega\,(n + \frac{1}{2})$ $\quad n = 0,1,2\ldots$

$|n\rangle = u_n(x) = N_n H_n(\xi)\,e^{-\frac{1}{2}\xi^2}$ where $\xi = \sqrt{\frac{m\omega}{\hbar}}\,x$ and H_n is an n^{th} degree polynomial

$$a_- = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p}{m\omega}\right) = \frac{1}{\sqrt{2}}\,(\xi + \partial_\xi) \qquad a_+ = a_-^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - i\frac{p}{m\omega}\right) = \frac{1}{\sqrt{2}}\,(\xi - \partial_\xi)$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-) \qquad p = i\sqrt{\frac{\hbar m\omega}{2}}\,(a_+ - a_-) \qquad a_-\,|n\rangle = \sqrt{n}\,|n-1\rangle \qquad a_+\,|n\rangle = \sqrt{(n+1)}\,|n+1\rangle$$

$$[a_-, a_+] = 1 \qquad [H, a_-] = -\hbar\omega a_- \qquad [H, a_+] = \hbar\omega a_+ \qquad H = \hbar\omega\left(\frac{1}{2} + a_+a_-\right)$$

Delta Function with $V(x) = -\alpha\delta(x) \Longrightarrow \Delta\psi' = -\frac{2m\alpha}{\hbar^2}\,\psi(0)$

$$u(x) = \sqrt{\kappa}\,e^{-\kappa|x|} \quad \text{where:} \quad \kappa = m\alpha/\hbar^2 \qquad E = -\frac{\kappa^2\hbar^2}{2m}$$

$$\psi(x) = \begin{cases} e^{ikx} + Re^{-ikx} & x < 0 \\ Te^{ikx} & x > 0 \end{cases} \qquad \text{where} \qquad R = \frac{-1}{ik/\kappa + 1}$$

Note: $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}\,dk = \frac{d}{dx}\Theta(x)$ $\qquad f(x_0) = \int f(x)\,\delta(x - x_0)\,dx$