$$\int_{-\infty}^{+\infty} e^{-\alpha x^2-\beta x}\,dx = \sqrt{\frac{\pi}{\alpha}}\,e^{\beta^2/4\alpha} \qquad \int_{-\infty}^{+\infty} x^2 e^{-\alpha x^2}\,dx = \frac{-d}{d\alpha}\int_{-\infty}^{+\infty} e^{-\alpha x^2}\,dx \qquad \int_0^\infty x^n\,e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}} \qquad \int f(x)\delta(x-a)\,dx = f(a)$$

$$H\psi = i\hbar\partial_t\psi \qquad H\psi = E\psi \qquad H = \frac{p^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\,\partial_x^2 + V(x) \qquad p = -i\hbar\partial_x \qquad [p,x] = -i\hbar$$

Notation: $Hu_n = E_n\,u_n$, $u_n(x)$ orthonormal, and $\omega_n \equiv E_n/\hbar$: $\Psi(x,t=0) = \sum c_n u_n(x) \Rightarrow \Psi(x,t) = \sum c_n u_n e^{-iw_n t}$

where (Fourier's Trick): $c_n = \langle u_n|\Psi\rangle$ Notation: $\langle f|Q|g\rangle \equiv \int_{-\infty}^{+\infty} f^*(x)\,Qg(x)\,dx = \int_{-\infty}^{+\infty} \left(Q^\dagger f(x)\right)^* g(x)\,dx = \langle Q^\dagger f|g\rangle$

$\partial_t\,\psi^*(x,t)\psi(x,t) = -\partial_x J$ where current $J = \frac{\hbar}{2im}\left(\psi^*\,\partial_x\psi - \psi\,\partial_x\psi^*\right) = \frac{\hbar}{2im}\,\psi^*\,\overleftrightarrow{\partial_x}\,\psi$

$$\frac{d}{dt}\langle\psi|A|\psi\rangle = \langle\psi|\partial_t A|\psi\rangle + \langle\psi|i[H,A]/\hbar|\psi\rangle \qquad [A,B] \equiv AB - BA$$

$$\sigma_A\sigma_B = \Delta A\,\Delta B \geq \frac{1}{2}\,|\langle\psi|i[A,B]|\psi\rangle| \qquad [AB,C] = A[B,C] + [A,C]B$$

Free Particle: $\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\,e^{ipx/\hbar}$ or $\psi_k(x) = \frac{1}{\sqrt{2\pi}}\,e^{ikx}$ where $p = \hbar k$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ikx}\,g(k)\,dk \qquad g(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ikx}\,f(x)\,dx$$

Particle-in-a-box with $V(x) = 0$ for $0 < x < L$, but $V(x) = \infty$ elsewhere

$$E_n = \frac{(\hbar k)^2}{2m} \qquad \psi_n(x) = \sqrt{\frac{2}{L}}\,\sin(kx) \quad \text{where} \quad k = \frac{n\pi}{L} \qquad n = 1,2,3\ldots$$

3-d: $|n_x n_y n_z\rangle = \psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z)$ $E = \frac{(\hbar k)^2}{2m}$ where $\vec{\mathbf{k}} = (n_x\pi/L_x, n_y\pi/L_y, n_z\pi/L_z)$

spherical box: $R(r) = j_\ell(kr)$ $E = \frac{(\hbar k)^2}{2m}$ where: kR = zero of j_ℓ

Harmonic Oscillator with $V(x) = \frac{1}{2}\,m\omega^2 x^2$ $E_n = \hbar\omega\,(n+\frac{1}{2})$ $n = 0,1,2\ldots$

$|n\rangle = \psi_n(x) = N_n H_n(\xi)\,e^{-\frac{1}{2}\xi^2}$ where $\xi = \sqrt{\frac{m\omega}{\hbar}}\,x$ and H_n is an n^{th} degree polynomial

$$a_- = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p}{m\omega}\right) = \frac{1}{\sqrt{2}}\,(\xi + \partial_\xi) \qquad a_+ = a_-^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - i\frac{p}{m\omega}\right) = \frac{1}{\sqrt{2}}\,(\xi - \partial_\xi) \qquad x = \sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-)$$

$$[a_-,a_+] = 1 \qquad [H,a_\pm] = \pm\hbar\omega a_\pm \qquad H = \hbar\omega\left(\tfrac{1}{2} + a_+a_-\right) \qquad a_-\,|n\rangle = \sqrt{n}\,|n-1\rangle \qquad a_+\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

2-d: $(n_x + n_y + 1) = (2n_r + |m| + 1)$ 3-d: $\left(n_x + n_y + n_z + \frac{3}{2}\right) = \left(2n_r + \ell + \frac{3}{2}\right)$

Angular Momentum: $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$ $[L_i, V_j] = i\hbar\epsilon_{ijk}V_k$ where vector $\vec{\mathbf{V}} = \vec{\mathbf{r}}, \vec{\mathbf{p}}, \vec{\mathbf{L}}$ $|\ell m\rangle = Y_{\ell m}(\theta,\phi)$ $-\ell \leq m \leq +\ell$

$$\vec{\mathbf{L}}^2\,|\ell m\rangle = \ell(\ell+1)\,\hbar^2\,|\ell m\rangle \qquad L_z\,|\ell m\rangle = m\hbar\,|\ell m\rangle \qquad L_\pm\,|\ell m\rangle = \sqrt{\ell(\ell+1) - m(m\pm 1)}\;\hbar\,|\ell\,m\pm 1\rangle$$

$$L_\pm = L_x \pm iL_y \qquad [L_+,L_-] = 2\hbar L_z \qquad [L_z,L_\pm] = \pm\hbar L_\pm \qquad [\vec{\mathbf{L}}^2, L_\pm] = 0 \qquad [L_i, \vec{\mathbf{V}}\cdot\vec{\mathbf{W}}] = 0$$

Spin $\frac{1}{2}$: $\vec{\mathbf{S}} = \frac{\hbar}{2}\,\vec{\sigma}$ $|\frac{1}{2}\,\frac{1}{2}\rangle = \chi_+ = \uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|\frac{1}{2}\,-\frac{1}{2}\rangle = \chi_- = \downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Clebsch-Gordan: $|jm\rangle = \sum C(jm;\ell m_\ell, sm_s)\,|\ell m_\ell\rangle\,|sm_s\rangle$ know how to use table!

Radial Equation: $\psi(r,\theta,\phi) = Y_{\ell m}(\theta,\phi)R(r)$ $R(r) = \frac{u(r)}{r}$

$$\left[\frac{-\hbar^2}{2m}\left(\partial_r^2 + \frac{2}{r}\partial_r\right) + \frac{\hbar^2\ell(\ell+1)}{2mr^2} + V(r)\right]R = E\,R \qquad \left[\frac{-\hbar^2}{2m}\partial_r^2 + \frac{\hbar^2\ell(\ell+1)}{2mr^2} + V(r)\right]u = E\,u$$

H atom: $H = \frac{p^2}{2m} - \frac{Ze^2}{4\pi\epsilon_0 r}$ $E_n = -\frac{1}{2}\,mc^2\,\frac{(Z\alpha)^2}{n^2} = -\frac{1}{2}\,\frac{Z^2e^2}{4\pi\epsilon_0 a_0 n^2} \approx -13.6\text{ eV}\,\frac{Z^2}{n^2}$ $n = 1,2,3,\ldots$

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2} \approx .53\text{ Å} \qquad \alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137} \qquad n = n_r + \ell + 1 \;\therefore\; 0 \leq \ell \leq n-1 \qquad \rho = \sqrt{\frac{8m|E|}{\hbar^2}}\,r = \frac{2Zr}{na_0}$$

$$|n\ell m\rangle = R_{n\ell}(\rho)\,Y_{\ell m}(\theta,\phi) \quad \text{where} \quad R_{n\ell} = N_{n\ell}\,\rho^\ell\,L_{n_r}^{2\ell+1}(\rho)\,e^{-\frac{1}{2}\rho} \qquad N_{n\ell} = \frac{2}{n^2}\sqrt{\frac{(n-\ell-1)!}{(n+\ell)!}}$$

2-particle CM Coordinates:

$$\vec{\mathbf{R}} = \frac{m_1}{M}\,\vec{\mathbf{r}}_1 + \frac{m_2}{M}\,\vec{\mathbf{r}}_2 \qquad \vec{\mathbf{r}}_1 = \vec{\mathbf{R}} + \frac{m_2}{M}\,\vec{\mathbf{r}} \qquad M = m_1 + m_2$$

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2 \qquad \vec{\mathbf{r}}_2 = \vec{\mathbf{R}} - \frac{m_1}{M}\,\vec{\mathbf{r}} \qquad \mu = \frac{m_1 m_2}{M}$$

$$\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} = \frac{P^2}{2M} + \frac{p^2}{2\mu}$$

Magnetic: $\vec{\mathbf{p}} \to \vec{\mathbf{p}} - q\vec{\mathbf{A}}$ where q is charge, e.g., for electron: $q = -e$ $\vec{\mathbf{B}} = \vec{\nabla} \times \vec{\mathbf{A}}$ e.g., uniform $\vec{\mathbf{B}}$ from $\vec{\mathbf{A}} = \frac{1}{2}\,\vec{\mathbf{B}} \times \vec{\mathbf{r}}$

Approximation Methods:

WKB: $\int k(x)\,dx = (n - \frac{1}{2})\pi$ (two linear turning points) $\hbar k(x) = p(x) = \sqrt{2m(E - V(x))}$

Rayleigh-Ritz: minimize $E = \dfrac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$

Perturbation Theory: $E_n^1 = \langle n|H'|n\rangle$ $E_n^2 = \sum_{k\neq n} \dfrac{|\langle k|H'|n\rangle|^2}{E_n^0 - E_k^0}$ degenerate: diagonalize matrix $\langle i|H'|j\rangle$

Time Dependent: $c_b(t) \simeq -\dfrac{i}{\hbar}\int_0^t H'_{ba}(t')\, e^{i(E_b - E_a)t'/\hbar}\, dt'$

if $H' = V(\mathbf{r})\cos\omega t$ then: $P_{a\to b} \approx \dfrac{|V_{ab}|^2}{\hbar^2}\,\dfrac{\sin^2((\omega_0 - \omega)t/2)}{(\omega_0 - \omega)^2}$ E1 Unpolarized Light: $R_{a\to b} = \dfrac{\pi\rho(\omega_0)}{3\epsilon_0\hbar^2}\,|q\langle\psi_b|\vec{\mathbf{r}}|\psi_a\rangle|^2$

Selection Rules: $\Delta m = \pm1, 0$ $\Delta l = \pm1$

E1: $\Delta J = 0, \pm1\ (0 \not\to 0)$, $\Delta M = 0, \pm1\ (0 \not\to 0,$ if $\Delta J = 0)$, $\Delta S = 0$, $\Delta L = 0, \pm1\ (0 \not\to 0)$

Spectroscopic Notation: orbital: s,p,d,f,g term: ${}^{2S+1}L_J$ atomic: 1s,2s,2p,3s,3p,4s,3d nuclear: 1s,1p,1d,2s,1f,2p,1g

Spin-statistics: fermion: $s = \frac{1}{2}, \frac{3}{2}, \ldots$ boson: $s = 0, 1, 2, \ldots$

$$u(x_i) = (N!)^{-1/2}\begin{vmatrix} f(x_1) & f(x_2) & f(x_3) & \cdots \\ g(x_1) & g(x_2) & g(x_3) & \cdots \\ h(x_1) & h(x_2) & h(x_3) & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix}$$

Scattering: $\dfrac{d\sigma}{d\Omega} = \dfrac{\text{hits/sec in detector}}{J d\Omega} = \dfrac{\text{hits/sec in detector}}{nt \text{ beam current } d\Omega} = \dfrac{b}{\sin\theta}\left|\dfrac{db}{d\theta}\right| = |f|^2$ where: $\psi \approx e^{ikz} + f(\theta)\,\dfrac{e^{ikr}}{r}$

$f(\theta) = \dfrac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin(\delta_\ell)P_\ell(\cos\theta)$ $\sigma = \dfrac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2(\delta_\ell)$ $f_{\text{Born}} = \dfrac{-m}{2\pi\hbar^2}\int e^{i\vec{\mathbf{q}}\cdot\vec{\mathbf{r}}_0}V(\vec{\mathbf{r}}_0)\, d^3\mathbf{r}_0$ $\vec{\mathbf{q}} = \vec{\mathbf{k}}_i - \vec{\mathbf{k}}_f$

$f_{\text{Born}} = f_{\text{Ruth}}F(q)$ where Form Factor: $F(q) = \int e^{i\vec{\mathbf{q}}\cdot\vec{\mathbf{r}}}\rho(r)\, d^3\mathbf{r}$ using normalized charge density: $\rho \ni$ $1 = \int \rho(r)\, d^3\mathbf{r}$

$F(k) \approx 1 - \dfrac{1}{6}k^2\langle r^2\rangle$ scattering length a: $\delta_0 \approx n\pi - ak$ effective range r_e: $\cot\delta_0 \approx \dfrac{-1}{ka} + \dfrac{1}{2}\, r_e k$