Starting from the usual force equation:

$$\ddot{r} = -\,G\frac{4}{3}\pi\rho r \tag{1}$$

the energy density of blackbody light:

$$\rho_\gamma = u/c^2 = \frac{4\sigma T^4}{c^3} \tag{2}$$

the usual scaling: $r = uR$ and how the light's temperature is affected by expansion:

$$T \longrightarrow \frac{T_0}{R} \tag{3}$$

we have:

$$\ddot{R} = -\frac{G\frac{4}{3}\pi\rho_\gamma}{R^3} \tag{4}$$

$$\frac{1}{2}\dot{R}^2 = \frac{G\frac{4}{3}\pi\rho_\gamma}{2R^2} \tag{5}$$

where an integration constant has been taken as zero. If we include radiation pressure, this result is doubled. Combining this result with previous:

$$\dot{R}^2 = \frac{G\frac{8}{3}\pi\rho_\gamma}{R^2} + \frac{G\frac{8}{3}\pi\rho_m}{R} + \frac{1}{3}\Lambda c^2 R^2 \tag{6}$$

where ρ_m and ρ_γ are the NOW ($R = 1$) values of mass and radiation density. Using the usual definitions

$$H_0 = 67.9 \text{ km/s/Mpc} \tag{7}$$

$$T_0 = 2.725 \text{ K} \tag{8}$$

$$\Omega_\gamma = \frac{8\pi G\rho_\gamma}{3H_0^2} = \frac{8\pi G 4\sigma T_0^4}{3c^3 H_0^2} = 5.4\times 10^{-4} \quad \text{adding neutrinos } \rightarrow 8.4\times 10^{-4} \tag{9}$$

$$\Omega_m = \frac{8\pi G\rho_m}{3H_0^2} = .306 \tag{10}$$

$$\Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2} = .694 \tag{11}$$

we get the general differential equation

$$\dot{R}^2 = H_0^2\left(\frac{\Omega_\gamma}{R^2} + \frac{\Omega_m}{R} + \Omega_\Lambda R^2\right) \tag{12}$$

In the early universe (say $R < .01$) we neglect the last term. Separating the resulting differential equation

$$
\begin{aligned}
\int_0^R \frac{dR}{\sqrt{\frac{\Omega_\gamma}{R^2}+\frac{\Omega_m}{R}}} &= H_0 t && (13)\\
\int_0^R \frac{R\,dR}{\sqrt{\Omega_\gamma+\Omega_m R}} &= H_0 t && (14)\\
\frac{2}{\Omega_m^2}\left[\frac{1}{3}X^{3/2}-\Omega_\gamma X^{1/2}\right]_0^R &= H_0 t && (15)\\
\frac{2}{\Omega_m^2}\left\{\frac{1}{3}\left(\sqrt{\Omega_\gamma+\Omega_m R}\right)^3-\Omega_\gamma\sqrt{\Omega_\gamma+\Omega_m R}+\frac{2}{3}\Omega_\gamma^{3/2}\right\} &= H_0 t && (16)\\
\frac{R^2}{2\sqrt{\Omega_\gamma}}-\frac{\Omega_m R^3}{6\Omega_\gamma^{3/2}}+\mathcal{O}(R^4) &= H_0 t && (17)
\end{aligned}
$$

In the last line the result has been series expanded to clearly show for early times

$$R \approx \sqrt{2\sqrt{\Omega_\gamma}H_0 t} \propto t^{1/2} \tag{18}$$

```
Integrate[r/Sqrt[OmegaG+OmegaM r ],{r,0,R},GenerateConditions->False]
Series[%,{R,0,3}]
                2                  3
              R             OmegaM R         4
Out[4]= -------------- - ----------- + O[R]
        2 Sqrt[OmegaG]             3/2
                            6 OmegaG

out=%%/H0 /. {OmegaM->.306,H0->1/14.4,OmegaG->.000084}
FindRoot[out==.00038,{R,.001}]
```

The last line allows finding R when $t = .00038$ (i.e., 380,000 years). Find z and $T = T_0/R$ at that time. This should be an improvement over the last homework result. Note from the choice of units for H_0 we are using time in billions of years.

Matter and light carry equal weight when

$$\frac{\Omega_\gamma}{R^2} = \frac{\Omega_m}{R} \tag{19}$$

When is that?

$t = 3$ minutes is the time of nuclear synthesis; find R and T at that time.