Torsion Pendulum

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This experiment used a torsion pendulum apparatus to calculate the damping constant, β, at points where the system is under dampened and critically dampened. This experiment also used a forcing motor to drive the pendulum at frequencies near resonance for three different brake currents and used the calculated resonance frequencies to determine β for each case. In this experiment, we determined that β is not constant for free rotational oscillations but is for forced rotational oscillations.

I. INTRODUCTION

This experiment[1] examines the behavior of a torsion pendulum, a special case among mechanical oscillators.

Note that angular frequency (ω in rad/s) and frequency (f in Hz.) are not the same. In the damped case, the torque balance for the torsion pendulum yields the differential equation[2]:

$$J\,\frac{d^2\theta}{dt^2} + b\,\frac{d\theta}{dt} + c\,\theta = 0 \tag{1}$$

where J is the moment of inertia of the pendulum, b is the damping coefficient, c is the restoring torque constant, and θ is the angle of rotation. This equation can be rewritten in the standard form[3]:

$$\ddot{\theta} + 2\beta\,\dot{\theta} + \omega_0^2\,\theta = 0 \tag{2}$$

where the damping constant is $\beta = b/2J$ and the natural frequency is $\omega_0 = \sqrt{c/J}$. The general solution to this differential equation is:

$$\theta(t) = e^{-\beta t}\left[A_1 e^{\sqrt{\beta^2-\omega_0^2}\,t} + A_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right] \tag{3}$$

with three different types of solutions possible depending on the relationships between ω_0 and β. In the underdamped case ($\beta < \omega_0$):

$$\theta(t) = \theta_0 e^{-\beta t}\cos(\omega_1 t - \gamma) \tag{4}$$

with the oscillation frequency $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$, initial amplitude θ_0, and phase γ. In the critically damped case ($\beta = \omega_0$):

$$\theta(t) = (A + Bt)e^{-\beta t} \tag{5}$$

In the overdamped case ($\beta > \omega_0$):

$$\theta(t) = e^{-\beta t}\left[A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t}\right] \tag{6}$$

where $\omega_2 = \sqrt{\beta^2 - \omega_0^2}$

For the forced oscillation case, an external torque is added to Eq. (1):

$$J\,\frac{d^2\theta}{dt^2} + b\,\frac{d\theta}{dt} + c\,\theta = \tau_0\sin(\omega t) \tag{7}$$

where ω is the driving frequency and τ_0 is the driving torque[4]. The general solution to the differential equation is the sum of the homogeneous solutions (which are the solutions to the damped case above) plus a particular solution. The particular solution has the form:

$$\theta(t) = \theta_m(\omega)\sin(\omega t - \phi) \tag{8}$$

with

$$\theta_m(\omega) = \frac{\tau_0}{J\sqrt{(\omega_0 - \omega)^2 + (b\omega/J)^2}} \tag{9}$$

In this case the resonance frequency is

$$\omega_r = \sqrt{\omega_0^2 - 2\beta^2} \tag{10}$$

and the phase shift between the pendulum and the external oscillator is:

$$\tan\phi = \frac{2\beta\omega}{\omega_0^2 - \omega^2} \tag{11}$$

II. THE EXPERIMENT

The experiment was conducted with the torsion pendulum apparatus set up as shown in Fig. 1, using a low voltage power supply and two digital multimeters.

For the first half of the experiment, the driving motor was not active.

We first determined was the period of the torsion pendulum to find the natural frequency by taking 3 measurements of 10 oscillations of the pendulum with the damping magnet turned off.

Next, we determined the damping constant, β, at two values of the damping current, 0.1 A $< I_1 <$ 0.3 A and 0.3 A $< I_2 <$ 0.6 A. First, we calculated the period of oscillation at each current and then started the pendulum at its furthest point of rotation and measured the amplitude after each period. We then fit the amplitude

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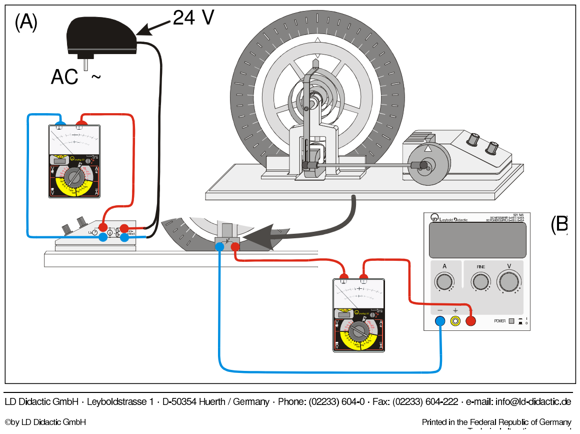


FIG. 1: A schematic of the torsion pendulum apparatus[4].

vs. time data to an exponential curve and used Eq. (4) to determine the damping constant.

The damping current was increased until the pendulum was near critically dampened, or when it only completed one oscillation after release from its furthest rotation point. For this case, we took the average oscillation time. We then further increased the current to the point where the pendulum was critically dampened, or did not cross zero after release from its furthest point of rotation. We used the critically dampened case to estimate the damping constant from Eq. (5) by measuring the time it took the pendulum to reach 1/10 of its initial value after being released from its furthest rotational point. In the critically dampened case, $B = 0$, A =furthest rotational point value at time $t = 0$. We used this β to get estimates of the damping coefficient, b, and restoring torque, c, where the moment of inertia of the pendulum.

The latter part of this experiment was conducted with the forcing motor turned on.

To determine a relationship between voltage and frequency of the forcing motor we determined the frequency of the motor at various voltages by finding the average period of the motor at several voltages and making a fit of frequency to obtain the relation between them. The resonance frequency curve for currents of, $I = 0$ A, $I \sim 0$ A, $I \sim 0.4$ A, $I \sim 0.8$ A were determined by taking measurements of amplitude of oscillations in intervals of 0.005 Hz from $f = 0.465$ Hz to $f = 0.560$ Hz for each damping current by varying the voltage of the motor accordingly.

III. DATA PRESENTATION AND INTREPRETATION OF RESULTS

Determination of the natural angular frequency, ω_0, with the damping current at $I = 0$ A:

Mean period at damping current $I = 149 \pm 3$ mA: 1.927 ± 0.005 seconds.

TABLE I: Results: the period, natural frequency and natural angular frequency of the torsion pendulum for braking $I = 0$ A.

Mean period for 10 oscillations	1.927 ± 0.005 s
Natural frequency	0.519 ± 0.0012 Hz
Natural angular frequency, ω_0	3.260 ± 0.008 rad/s

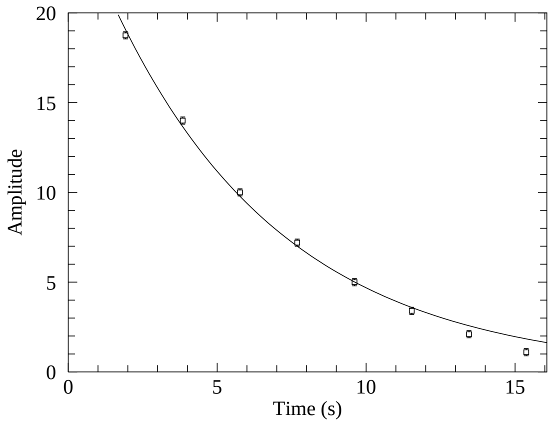


FIG. 2: Plot of Amplitude vs. time fit to an exponential, where each data point is spaced apart by one period (1.927 ± 0.007 seconds)

The data for amplitude vs. time for a damping current of $I = 149 \pm 3$ mA were taken at each consecutive period starting after the pendulum had gone through one oscillation. The uncertainty in measurement of amplitude was assumed to be 0.2 at each measurement with an uncertainty in time given by: $\delta t = |n|\delta T$. When these data were fit to an exponential curve, the fit equation was: $y = Ae^{Bx}$, where $A = 21.2 \pm 0.5$ and $B = -0.040 \pm 0.001$ rad/s. β was found to be $-B$ as per Eq. (4): $\beta = 0.040 \pm 0.001$ rad/s.

For a damping current of 490 ± 7 mA, the mean period was found to be 1.927 ± 0.007 seconds

The data of amplitude vs. per consecutive period with a damping current of 490 ± 7 mA were taken in a similar fashion. Again, the uncertainty in amplitude was taken to be 0.2 and uncertainty in time was : $\delta t = |n|\delta T$. These data were also fit to an exponential curve as seen in Fig. 2 with a reduced chi-squared of 4.7. The equation of the fit line was of the form $y = Ae^{Bx}$ where $A = 26.6 \pm 0.4$ and $B = -0.17 \pm 0.03$ rad/s which, as per Eq. (4), gives the value of the damping constant to be $\beta = 0.17 \pm 0.03$ rad/s, a reasonable answer as one would expect the damping constant to increase as the damping current increases.

The system was near critical dampening at a damping current of 1.69 ± 0.02 A with a mean stopping time of 2.3 ± 0.1 seconds. We found that the system was critically dampened at a damping current of 1.90 ± 0.02 A with a mean stopping time of 1.6 ± 0.3 seconds. The results for the time to stop agree with the idea that the pendulum comes to rest at sooner when the damping is increased

For the critically dampened case, the mean time for θ to decrease from its furthest point of rotation $\theta_0 = 19$ to $\theta = 1.9$ was found to be 1.00 ± 0.08 seconds. We used this in conjunction with Eq. (5) to get estimates of β, b, and c. The damping constant was found using following equation:

$$\beta = \frac{-\ln(\theta/\theta_0)}{\Delta t} \tag{12}$$

and uncertainty in β was

$$\delta\beta = \sqrt{\left(\frac{\partial\beta}{\partial\theta}\delta\theta\right)^2 + \left(\frac{\partial\beta}{\partial\theta_0}\delta\theta_0\right)^2 + \left(\frac{\partial\beta}{\partial\Delta t}\delta\Delta t\right)^2} \tag{13}$$

$$= \frac{1}{\Delta t}\sqrt{\left(\frac{\delta\theta}{\theta}\right)^2 + \left(\frac{\delta\theta_0}{\theta_0}\right)^2 + (\beta\,\delta\Delta t)^2} \tag{14}$$

where Δt =the mean time for θ to decrease from θ_0. The estimate the damping coefficient, b, was found by from $b = 2J\beta$ and

$$\delta b = \sqrt{\left(\frac{\partial b}{\partial J}\,\delta J\right)^2 + \left(\frac{\partial b}{\partial\beta}\,\delta\beta\right)^2} \tag{15}$$

where $J = 3.0 \pm 0.1$ kg $\cdot$ m^2. An estimation of the restoring torque was found from $c = J\beta^2$ and

$$\delta c = \sqrt{\left(\frac{\partial c}{\partial J}\,\delta J\right)^2 + \left(\frac{\partial c}{\partial\beta}\,\delta\beta\right)^2} \tag{16}$$

See Table II for the obtained values. In the critically dampened case, it is expected that $\beta = \omega_0$, which is not exactly the case as seen in comparing β from Table II and ω_0 from Table I.

TABLE II: Results: estimations of β, b, and c for a near critically dampened torsion pendulum (braking $I = 1.90 \pm 0.2$ A).

$\beta = 2.3 \pm 0.2$ rad/s
$b = 14 \pm 1$ kg $\cdot$ m$^2 \cdot$ s^{-1}
$c = 16 \pm 3$ N $\cdot$ m

The latter part of this experiment dealt with forced oscillations of the pendulum.

A relationship between frequency and voltage was established by determining the mean period and thus frequency at a range of dial settings along with the voltage across the motor measured by the multimeter and fitting that data to a linear function.

The equation relating frequency and voltage is of linear of the form $y = A + Bx$, where $A = -0.01 \pm 0.006$ Hz, $B = (0.071 \pm 0.11) \times 10^{-2}$ Hz/V, y is the frequency and x is the voltage across the motor.

TABLE III: Resonance frequencies and the associated angular resonance frequencies, in addition to the calculated damping constant at each brake current for the driven oscillator.

braking I, mA	0	71 ± 2	424 ± 6	830 ± 10
f_r, Hz	0.528 $\pm$0.003	0.527 $\pm$0.003	0.523 $\pm$0.004	0.510 $\pm$0.007
ω_r, rad/s	3.34 $\pm$0.02	3.31 $\pm$0.02	3.29 $\pm$0.03	3.20 $\pm$0.04
β, rad/s	0.51 $\pm$0.07	0.41 $\pm$0.09	0.3 $\pm$0.1	0.4 $\pm$0.1

We then chose the range of frequencies to look at resonance to be 0.465 Hz $\leq f \leq 0.560$ Hz for each damping current. Measurements of amplitude vs. frequency were taken by setting the forcing motor to the particular voltage to give the desired frequency, stopping the pendulum before each measurement and letting the pendulum settle for a few minutes for each measurement.

Figure 3 displays the resonance curves for braking currents $I = 0$ A, $I = 424 \pm 6$ mA, and $I = 0.83 \pm 0.01$ A. (The resonance curve for braking $I = 72 \pm 2$ mA was indistinguishable from the resonance curve for $I = 0$ A.) The plots were used to determine the resonance frequency. The uncertainties in resonance frequency were estimated based on accuracy to which we could determine the frequency of resonance for each braking current. Using Eq. (10), the damping constant was estimated by

$$\beta = \sqrt{\frac{\omega_0^2 - \omega_r^2}{2}} \tag{17}$$

and the uncertainty of β was given by

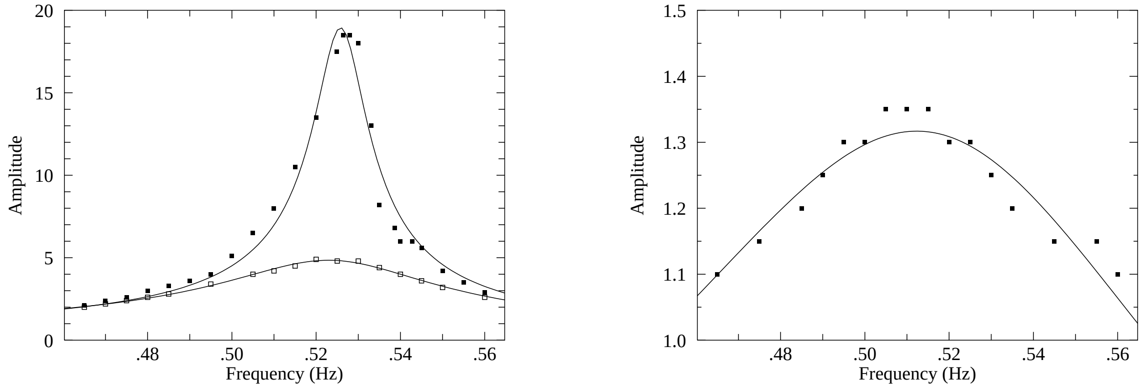
$$\delta\beta = \sqrt{\left(\frac{\partial\beta}{\partial\omega_r}\delta\omega_r\right)^2 + \left(\frac{\partial\beta}{\partial\omega_0}\delta\omega_0\right)^2} \tag{18}$$

with $\omega_0 = 3.26 \pm 0.02$ rad/s . The values of f_r , ω_r, and β obtained are shown in Table III below.

As expected, the resonance frequency differs less from the natural frequency with lower damping.

It can be seen in Fig. 4 that as the difference between the angular driving frequency and the natural angular frequency approaches zero, the phase difference between the pendulum and the forcing motor approaches $\pm\pi/2$, and at high and low frequencies, the phase approaches zero. The phase plotted in Fig. 4 was given by Eq. (11) with range properly shifted to 0 to *pi*.

Comparing the damping constants found for dampened and forced oscillation at similar values of damping current, as shown in Tables IV and V, it is evident that β is not consistent. This could be a result of the poor fit for the damping current of $I = 149 \pm 3$ mA. The damping



(a) Plots of the resonance curves for damping currents of $I = 0$ mA (■) and $I = 424 \pm 6$ mA (□)

(b) Plot of the resonance curve for a damping current of $I = 0.830 \pm 0.01$ A

FIG. 3: Plots of amplitude vs. frequency for various braking currents. The curves are fits to Eq. (9)

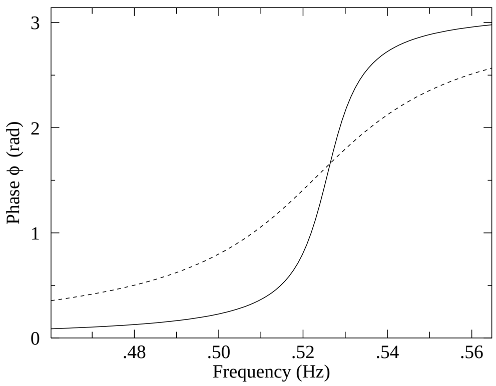


FIG. 4: A plot of phase vs. frequency for $I = 0$ mA (solid) and $I = 424 \pm 6$ mA (dashed) using Eq. (11) and the fit paramters used in the curves of Fig. 3a

TABLE IV: A comparison of β calculated for dampened oscillations at different damping currents.

braking I, mA	149 ± 3	490 ± 7	1900 ± 20
β, rad/s	0.040 ± 0.001	0.17 ± 0.03	2.3 ± 0.2

TABLE V: A comparison of β calculated using the resonance curve of forced oscillations at different braking currents.

braking I, mA	0	72 ± 2	424 ± 6	830 ± 10
β, rad/s	0.51 ± 0.07	0.41 ± 0.09	0.3 ± 0.1	0.4 ± 0.1

constant for damping currents of ~ 0.45 A are marginally close but do not agree within the bounds of uncertainty, which may again be a result of inaccuracies of the fit of the data.

[1] J.P. Crumley, *Torsion Pendulum*, `http://www.physics.csbsju.edu/~jcrumley/370/tp/torsion_pendulum.pdf`
[2] Leybold Scientific, *Free Rotational Oscillations*, `http://www.leybold-didactic.de/literatur/hb/e/p1/p1531_e.pdf`, (LD Didactic, Huerth, Germany, 2006)
[3] S.T. Thornton and J.B. Marion, *Classical Dynamics of Particles and Systems*, 5th Ed. (Brook/Cole-Thomson, Belmont, CA, 2003)
[4] Leybold Scientific, *Forced Rotational Oscillations*, `http://www.leybold-didactic.de/literatur/hb/e/p1/p1532_e.pdf`, (LD Didactic, Huerth, Germany, 2006)