Basic Optics: Radiance

Astronomy 525

Lecture 06

Outline

- The Radiance Theorem
- Basic Radiance
- Abbe’s Sine Condition
- Étendue
- Plate scales: re-imaging of pixel

**Radiance Theorem**

Let $d^2\Phi$ be the power (Watts) emitted into a solid angle $d\Omega$ by a source of element of projected area $dA_{proj}$. Then, the radiance, $L$, is defined by:

$$L = \frac{d^2\Phi}{dA_{proj} \cdot d\Omega} \quad (W/m^2/sr)$$

where:

$$dA_{proj} = dA \cos \theta$$

**Units of intensity**

**Radiance Theorem: I**

The radiance is conserved through a loss-less optical system.

$$d\Omega_S = \text{solid angle of } dA_R \text{ at } dA_S = \frac{(dA_R \cdot \cos \theta_R)}{r^2} \quad (1)$$

$$d\Omega_R = \text{solid angle of } dA_S \text{ at } dA_R = \frac{(dA_S \cdot \cos \theta_S)}{r^2} \quad (2)$$

The power, $d^2\Phi$ transferred from $dA_S$ to $dA_R$ is:

$$d^2\Phi = L_S \cdot (dA_S \cos \theta_S) \cdot d\Omega_S \quad (3)$$

By the definition of the radiance, $L_S$
Radiance Theorem: II

The radiance, $L_R$, measured at $dA_R$ (in the same direction) is:

$$L_R = \frac{d^2\Phi}{dA_R \cos \theta_R \, d\Omega_R}$$

where $d\Omega_R$ is given by equation (2) above, since the flux leaves $dA_R$ in a solid angle equal to that from which it arrived.

Using equations (1), (2), and (3) above yields:

$$L_R = L_S$$

Radiance Theorem: III

As a side result, we can show that it is possible to adopt the point of view of either the source or receiver when performing radiometric calculations.

Consider:

$$d^2\Phi = L_S \{dA_S \cos \theta_S\} \, d\Omega_S$$

$$= L_S \cos \theta_S \, dA_S \{dA_R \cos \theta_R/r^2\}$$

$$= L_S \, dA_R \cos \theta_S \{dA_R \cos \theta_S/r^2\}$$

$$= L_S \{dA_R \cos \theta_R\} \, d\Omega_R$$

That is, we can think of the power we would measure in two ways:

1. From the source point of view: $d^2\Phi \propto dA_{proj}(source) \, d\Omega_S$ of receiver

2. From the receiver point of view: $d^2\Phi \propto dA_{proj}(receiver) \, d\Omega_R$ of source
Basic Radiance: I

Suppose we have a beam of radiance, \( L \), passing through a medium with refractive index \( N_1 \), falling onto \( dA \) from solid angle \( d\Omega_1 \) inclined at \( \theta_1 \) w.r.t. \( dA \), then the power passing through \( dA \) is given by:

\[
d^2\Phi = L_1 dA \cos \theta_1 d\Omega_1
\]

We would like to find \( \theta_2 \) and \( \Omega_2 \) in terms of find \( \theta_1 \) and \( \Omega_1 \). Using polar coordinates, with the axis normal to \( dA \), we have:

\[
d\Omega_1 / d\Omega_2 = (\sin \theta_1 d\theta_1 d\varphi_1) / (\sin \theta_2 d\theta_2 d\varphi_2)
\]

(Differentiating): \[N_1 \cos \theta_1 d\theta_1 = N_2 \cos \theta_2 d\theta_2\]

Using equation (1) above, we then have:

\[
d\Omega_1 / d\Omega_2 = (N_2/N_1)^2 (\cos \theta_2) / (\cos \theta_1)
\]

The radiance of the refracted beam is then:

\[
L_2 = d^2\Phi / (dA \cos \theta_2 d\Omega_2) = L_1 dA \cos \theta_1 \Omega_1 / (dA \cos \theta_2 \Omega_2) = L_1 (N_2/N_1)^2
\]

\[\Rightarrow L_1/(N_1)^2 = L_2/(N_2)^2\]

Basic Radiance: II

From Snell's law we have: \( d\varphi_1 = d\varphi_2 \) (pick your plane of propagation) and \( N_1 \sin \theta_1 = N_2 \sin \theta_2 \)

so that (differentiating):

\[N_1 \cos \theta_1 d\theta_1 = N_2 \cos \theta_2 d\theta_2\]

Using equation (1) above, we then have:

\[
d\Omega_1 / d\Omega_2 = (N_2/N_1)^2 (\cos \theta_2) / (\cos \theta_1)
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The radiance of the refracted beam is then:

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L_2 = d^2\Phi / (dA \cos \theta_2 d\Omega_2) = L_1 dA \cos \theta_1 \Omega_1 / (dA \cos \theta_2 \Omega_2) = L_1 (N_2/N_1)^2
\]

\[\Rightarrow L_1/(N_1)^2 = L_2/(N_2)^2\]
Basic Radiance: III

Intuitive Proof of the Basic Radiance Theorem:
\[ d^2\Phi = L_1 \, d\Omega_1 \, dA = L_2 \, d\Omega_2 \, dA \]
small angles on axis: \( \sin \theta \approx 0 \Rightarrow d\Omega \propto \theta^2 \)
\[ \Rightarrow L_1 \, \theta_1^2 \, dA \approx L_2 \, \theta_2^2 \, dA \]
Snell's Law: \( N_1 \theta_1 \approx N_2 \theta_2 \Rightarrow N_1/N_2 \approx \theta_2/\theta_1 \)
\[ L_1(\theta_1/\theta_2)^2 = L_2 \]
or:
\[ L_1(N_2/N_1)^2 = L_2 \]

Abbe's Sine Condition: I

Suppose we have a source of height \( h_1 \) in medium of index \( N_1 \), imaged into medium \( N_2 \). We will show that the image height is related to the source height by:
\[ N_1 h_1 \sin \theta_1 = N_2 h_2 \sin \theta_2 \]
Abbe’s Sine Condition: II

Thermodynamic Proof:
By the radiance theorem, if the radiance of \( dA_1 \), measured in the medium with index \( N_1 \) is \( L_o(N_1)^2 \), then the radiance of \( dA_2 \) measured into its surrounding medium must be \( L_o(N_2)^2 \).

Now, \( dA_1 \) radiates a power \( d\Phi \) into an annular element of solid angle with half angle \( \alpha \) of:

\[
d\Phi = 2\pi L_o(N_1)^2 dA_1 \cos \alpha \sin \alpha \, d\alpha
\]

Radiance theorem/\( \Omega \)

Therefore, the total power transferred from \( dA_1 \) to \( dA_2 \) is:

\[
d\Phi_1 = \int_0^{\theta_1} d\Phi_1 = \pi L_o(N_1)^2 dA_1 \sin^2 \theta_1
\]

where \( \theta_1 \) is the half angle subtended by the aperture from the point of view of \( dA_1 \). Similarly, the power transferred from \( dA_1 \) to \( dA_2 \) is given by:

\[
d\Phi_2 = \int_0^{\theta_2} d\Phi_2 = \pi L_o(N_2)^2 dA_2 \sin^2 \theta_2
\]

By the second law of thermodynamics: \( d\Phi_1 = d\Phi_2 \)

\[
\Rightarrow \quad N_1 h_1 \sin \theta_1 = N_2 h_2 \sin \theta_2
\]

where \( h_1 \) and \( h_2 \) are the linear dimensions of \( A_1 \) and \( A_2 \).
Étendue: I

In general, the basic radiance, defined by: \( L/N^2 \) of a narrow beam of radiation is conserved as the beam propagates through any loss-less optical system. (see Boyd, section 5.2)

Total Power Measurement (Boyd, section 5.5)

What is the total power transmitted by a perfectly transmitting optical system (i.e. no vignetting, absorption, etc.)?

The power is given by:

\[
\Phi = \int \int L(r, n) \, dA \cos \theta \, d\Omega
\]

where \( L(r, n) \) is the source radiance of the point \( r \) in the direction of unit vector \( n \). The surface integral is over the entrance window, and the solid angle integral extends over the solid angle subtended by the entrance window.

Étendue: II

To characterize the properties of the optical system assume the source is uniform and Lambertian (\( L(n) = L_0 \)), then:

\[
\Phi = L_0 \int \int dA \cos \theta \, d\Omega
\]

\[
= L_0/(N_0)^2 \, \varepsilon
\]

where \( N_0 \) = the index of refraction, and

\( \varepsilon = \text{étendue} \) of the system

\[
= (N_0)^2 \int \int dA \cos \theta \, d\Omega
\]

The étendue is a purely geometric quantity that is a measure of the flux gathering capability of the optical system. The collected power is the product of \( \varepsilon \) and the basic radiance of the source.

\[
\text{power} = \text{étendue} \cdot \text{radiance}
\]

\[
\text{area} \cdot \text{solid angle} \quad \text{intensity (W/m}^2/\text{sr)}
\]
ändue: III

Consider the optical system above. Suppose $A_o$, the area of the source is small so that the solid angle subtended by the entrance pupil of the optical system does not change over the source. Then:

$$
\mathcal{E} = (N_o)^2 A_o \int \cos \theta \, d\Omega
$$
or:

$$
\mathcal{E} = (N_o)^2 A_o \Omega_{\text{proj},o}
$$

where we define the projected solid angle by:

$$
\Omega_{\text{proj},o} = \int \cos \theta \, d\Omega
$$

Éndue: IV

Since the entrance pupil subtends a half angle $\theta_o$,

$$
\Omega_{\text{proj},o} = \int_0^{\theta_o} 2\pi \sin \theta \cos \theta \, d\theta = \pi \sin^2 \theta_o
$$

and hence:

$$
\mathcal{E} = \pi (N_o)^2 A_o \sin^2 \theta_o
$$

Now, recall that $Nh \sin \theta$ (h = height of the object/image) is conserved between the object and image in a well corrected imaging system (Abbe’s sine condition). Therefore, the étendue is invariant between the image and object planes.
Étendue: V

More generally, consider an element of the étendue:

\[ d^2 \varepsilon = N^2 \, dA \cos \theta \, d\Omega \]

If we also consider the flux passing through the same element of area into the same solid angle, and in the same direction:

\[ d^2 \Phi = L \, dA \cos \theta \, d\Omega \]

Which yields:

\[ d^2 \Phi = \frac{L}{N^2} \cdot d^2 \varepsilon \]

Now, in any loss-less system, by conservation of energy, \( d^2 \Phi \) is conserved. We also have by the general form of the radiance theorem, that \( L/N^2 \) is invariant. Hence, \( d^2 \varepsilon \) must be invariant. We have then, that:

\[ \varepsilon = \int \int d^2 \varepsilon = N^2 \int \int dA \cos \theta \, d\Omega \]

is conserved. The étendue can be evaluated over any surface that intersects all the rays passing through the system.

Étendue: VI

The invariance of étendue forms the basis for our usual statement that:

\[ A\Omega = \text{constant} \]

in an optical system. Keep in mind, however, that this expression is not the most general form.

The fact that \( A\Omega \) is conserved provides a very powerful tool for optical system design.

Example:

\[ A_1 \Omega_1 = A_2 \Omega_2 \]

where \( A = \pi/4 \, d^2 \), and \( \Omega = \pi/4 \, \theta^2 \)

\[ \Rightarrow d_1 \, \theta_1 = d_2 \, \theta_2 \]

But, since \( f\# = f/D = 1/\theta \)

\[ \Rightarrow d_1/f\#_1 = d_2/f\#_2 \]
Plate Scale: I

Consider the case where we wish to match the image size from a telescope to the size of a pixel in a CCD camera.

For the Palomar telescope, with an f/15.7 secondary, the plate scale is given by:

\[ x_T = \frac{f_T}{D_T} \cdot \frac{\theta_s}{\text{radian}} \cdot 5000 \text{ mm} \cdot 15.7 \]

\[ = \frac{1^\prime}{206,265^\prime/\text{radian}} \cdot 5000 \text{ mm} \cdot 15.7 \]

\[ = 0.387 \text{ mm for } 1^\prime \]

\[ \Rightarrow \text{plate scale} = 2.6^\prime/\text{mm} \]

Thus, to cover 0.5" with a "pixel", we need a detector with is 193 μm across.

Plate Scale: II

Typical CCD's have pixels ~ 25 μm (x_d) across, so that we need to re-image to obtain the correct plate scale. From our relation \( A\Omega = \text{constant} \), we can easily determine the \( f'_c \) of the final optical stage (camera).

\[ f'_c = \frac{x_d}{x_T} \cdot f_T \]

\[ = \frac{25}{193} \cdot 15.7 \]

\[ = 2.0!! \text{ (a very fast camera!)} \]

Note:

\[ x_T = \theta_s \cdot \frac{f'_c}{D_T} \]

\[ \Rightarrow f'_c = \frac{\theta_s \cdot D_T}{x_T} \]

\( i.e., \text{ we get the desired } f' \text{ of the final camera in terms of the plate scale desired, and the primary aperture. } \)
Plate Scale: III

It is often useful to match the diffraction spot from the telescope to the size of the detector. This is relevant for diffraction limited (not seeing limited) observations. For diffraction from a filled aperture, the full width, at half maximum of the beam is given by:

$$\theta_{\text{diff}} = \frac{1.22 \lambda}{D_T}$$

Where D is the size of the primary mirror.

Therefore, since $d = \theta_{\text{f\#}} \cdot D$, we have

$$d_{\text{diff}} = \frac{1.22 \lambda}{D_T \cdot f_{\text{\#}}} \cdot D_T$$

$$= 1.22 \lambda \cdot f_{\text{\#}}$$

For the visible, $\lambda = 0.5 \, \mu\text{m}$:

$$d_{\text{diff}} = 1.22 \cdot 0.5 \cdot 2.0$$

$$= 1.2 \, \mu\text{m}$$

Therefore, the pixel size is large compared with the diffraction limit in this case.

Plate Scale: IV

At longer wavelengths, it is often possible to obtain the diffraction limited images. For example, the Hale 5 m telescope is diffraction limited at $\lambda > 10 \, \mu\text{m}$, and approaches the diffraction limit for $\lambda > 2 \, \mu\text{m}$ if adaptive optics or speckle techniques are used. To obtain the diffraction limit in a single exposure, one needs to sample the focal plane at the Nyquist frequency: 2 pixels per diffraction limited beam:

$$\theta_{\text{diff}} = \frac{1.22 \lambda}{D_T}$$

$$\Rightarrow f_{\text{\#}}^c = \frac{d_{\text{diff}}}{(\theta_{\text{diff}/2} \cdot D_T)}$$

$$\Rightarrow f_{\text{\#}}^c = \frac{d_{\text{diff}}}{(1.22 / 2 \lambda)}$$

"\lambda-f over 2 pixels"

Spectrocam-10 on the Hale 5 m telescope has 75 $\mu\text{m}$ square pixels, and is designed to fully sample the focal plane at 10 $\mu\text{m}$. The final $f_{\text{\#}}$ must therefore be:

$$f_{\text{\#}}^c = 75 \, \mu\text{m} / (0.61 \cdot 10 \, \mu\text{m}) = 4.5$$