Thermionic Emission

Introduction

Metals, as demonstrated by their ability to conduct an electric current, contain mobile electrons which are not strongly bound to individual atoms. (Most electrons in metals, particularly the “core” electrons closest to the nucleus, are tightly bound to individual atoms; it is only the outermost “valence” electrons that are somewhat “free”.) These free electrons are generally confined to the bulk of the metal. As you learned in E&M, an electron attempting to leave a conductor experiences a strong force attracting it back towards the conductor due to an image charge:

\[ F_x = -\frac{e^2}{4\pi\varepsilon_0(2x)^2} \]  

(1)

where \( x \) is the distance the electron is from the interface and \( e \) is the absolute value of the charge on an electron. Of course, inside the metal the electric field is zero so an electron there experiences zero (average) force. You can think of these valence electrons as bouncing around inside a box whose “walls” are provided by the image-charge force. (Odd to think: the “walls” are non-material force fields; the “inside” of the box is filled with solid metal.) Since temperature is a measure of random kinetic energy, if we increase the temperature of the metal, the electrons will be moving faster and some will have enough energy to overcome the image-charge force (which after all becomes arbitrarily small at large distances from the interface) and escape. This is just electron “evaporation”. The higher the temperature the larger the current of escaping electrons. This temperature induced electron flow is called thermionic emission. Starting in 1901, Owen Richardson studied this phenomenon and in 1929 he received the Nobel prize in Physics for his work. A hot wire will be surrounded by evaporated electrons. An electric force can pull these electrons away from the wire — the larger the electric force, the larger the resulting current of electrons. The precise relationship between the voltage and the resulting current flow is called Child’s law (or the Child-Langmuir law, including Langmuir who independently discovered it while working at G.E.). In this experiment you will measure both of these phenomena.

Child’s Law

Consider a planar interface between a metal \((x < 0)\) and “vacuum” \((x > 0)\). Electrons form a cloud of escaped electrons surrounding the hot metal filament (called the cathode, which we take to be at zero potential). Thus the “vacuum” adjacent to the metal is not empty, rather it is filled with electrons (with number density \(n(x)\)—i.e., a space charge). A positive potential, \(V_A\), on the anode plane located at \(x = b\) provides a force pulling these electrons from the vicinity of the cathode towards the anode. The electrons moving towards the right constitute a steady electric current density towards the left, i.e., a steady electric current from the anode to the cathode:

\[ J = -en(x)v(x) = -J_A \]  

(2)

where \(v(x)\) is the electron velocity. Since the electrons leave the metal with (nearly) zero speed at zero potential, we can calculate their speed along the path to the anode using
Figure 1: A planar cathode and a planar anode are separated by a distance $b$. A positive potential difference $V_A$ attracts electrons from the cathode to the anode, so the speed of the electrons $v(x)$ increases as they approach the anode. The moving electrons constitute an electric current from anode to cathode. The resulting steady current density is called $J_A$.

conservation of energy:

$$\frac{1}{2}mv^2 - eV(x) = 0$$

$$v = \sqrt{\frac{2e}{m} V(x)}$$

Because the accelerating electrons constitute a steady current (i.e., $J_A$ doesn’t depend on position), $n(x)$ must decrease as the electrons speed toward the anode. The varying space charge density affects the electric potential in the “vacuum” according to Poisson’s equation:

$$\frac{\partial^2 V}{\partial x^2} = -\frac{\rho(x)}{\epsilon_0} = \frac{en(x)}{\epsilon_0}$$

Putting these pieces together with have the differential equation:

$$\frac{d^2 V}{dx^2} = \frac{J_A}{\epsilon_0 v(x)} = \frac{J_A}{\epsilon_0 \sqrt{\frac{2e}{m} V(x)}}$$

Since the electric field will be zero at the interface, we have a pair of initial conditions:

$$\frac{\partial V}{\partial x} \bigg|_{x=0} = 0$$
$$V \bigg|_{x=0} = 0$$

This differential equation looks a bit like Newton’s second law:

$$\frac{d^2 x}{dt^2} = \frac{1}{m} F(x(t))$$

as you can see if in Newton’s second law you substitute:

$$\frac{1}{m} F(x(t)) \rightarrow \frac{J_A}{\epsilon_0 \sqrt{\frac{2e}{m} V(x)}}$$
Recall that force problems are often most simply solved using conservation of energy and that conservation of energy was proved using an integrating factor of $dx/dt$. If we try the analogous trick on our voltage problem, we’ll multiply Poisson’s equation by $dV/dx$:

$$\frac{dV}{dx} \times \frac{d^2V}{dx^2} = \frac{J_A}{\epsilon_0 \sqrt{\frac{2e}{m}}} V^{-\frac{1}{2}} \times \frac{dV}{dx}$$

(10)

$$\left( \frac{1}{2} \left[ \frac{dV}{dx} \right]^2 \right)' = \frac{J_A}{\epsilon_0 \sqrt{\frac{2e}{m}}} \left( \frac{V^{\frac{1}{2}}}{\epsilon_0 \sqrt{2e/m}} \right)'$$

(11)

$$\frac{1}{2} \left[ \frac{dV}{dx} \right]^2 = \frac{J_A}{\epsilon_0 \sqrt{\frac{2e}{m}}} \frac{V^{\frac{1}{2}}}{\epsilon_0 \sqrt{2e/m}} + \text{constant}$$

(12)

The initial conditions require the constant to be zero, so

$$\frac{1}{2} \left[ \frac{dV}{dx} \right]^2 = \frac{J_A}{\epsilon_0 \sqrt{\frac{2e}{m}}} \frac{V^{\frac{1}{2}}}{\epsilon_0 \sqrt{2e/m}}$$

(13)

or

$$\frac{dV}{dx} = \sqrt{\frac{4J_A}{\epsilon_0 \sqrt{\frac{2e}{m}}}} V^{\frac{1}{4}}$$

(14)

This differential equation is separable:

$$\frac{dV}{V^{\frac{1}{4}}} = \sqrt{\frac{4J_A}{\epsilon_0 \sqrt{\frac{2e}{m}}}} \, dx$$

(15)

$$V^{\frac{3}{4}} = \sqrt{\frac{4J_A}{\epsilon_0 \sqrt{\frac{2e}{m}}}} x$$

(16)

where again the initial conditions require the constant of integration to be zero. Finally:

$$V(x) = \left[ \frac{9J_A}{4\epsilon_0 \sqrt{\frac{2e}{m}}} \right]^{\frac{2}{3}} x^{\frac{4}{3}}$$

(17)

Of course, $V(b)$ is the anode voltage $V_A$, so we can rearrange this equation to show Child’s law:

$$J_A = \left[ \frac{4\epsilon_0 \sqrt{\frac{2e}{m}}}{9b^2} \right] V_A^{\frac{3}{4}}$$

(18)

Much of Child’s law is just the result of dimensional analysis, i.e., seeking any possible dimensionally correct formula for $J_A$. Our differential equation just involves the following constants with dimensions as shown:

$$b : L$$

$$V_A : \frac{E}{Q} = \frac{ML^2/T^2}{Q}$$

(19)
Figure 2: Coaxial cylinders: an inner cathode (radius $a$) and outer anode (radius $b$), form a vacuum tube diode. A potential difference $V_A$ attracts electrons from the cathode to the anode, so the speed of the electrons $v(r)$ increases as they approach the anode. The moving electrons constitute a steady electric current from anode to cathode. Since the same current is spread out over larger areas, the current density, $J$, between the cylinders must be proportional to $1/r$.

$$\epsilon_0 \sqrt{\frac{2e}{m}} \equiv k : \quad \frac{Q^2}{EL} \frac{Q_A^2}{M^2} = \frac{Q_A^2}{M^2 L^3 / T^2}$$  \quad (21)

$$J_A : \quad \frac{Q}{T L^2}$$  \quad (22)

To make a dimensionally correct formula for $J_A$, we just eliminate the $M$ dimension which we can only do with the combination:

$$V_A k^\frac{2}{3} : \quad \frac{Q_A^2}{T^\frac{2}{3}}$$  \quad (23)

We can then get the right units for $J_A$ with:

$$\left(\frac{V_A k^\frac{2}{3}}{b^2}\right)^{\frac{2}{3}} = k \frac{k}{b^2} V_A^\frac{2}{3} : \quad \frac{Q}{T L^2}$$  \quad (24)

Thus the only possible dimensionally correct formula is

$$J_A \propto \frac{k}{b^2} V_A^\frac{2}{3}$$  \quad (25)

The exact proportionality constant, found from the differential equation, is (as usual) is not hugely different from 1.

We have derived Child’s law for the case of infinite parallel plates, but you will be testing it in (finite length) coaxial cylinders. The inner cylinder (radius $a$) is the cathode; the outer cylinder (radius $b$) is the anode. Your cylinder will have some length $\ell$, but we will below consider infinite length coaxial cylinders. Note that dimensional considerations require that the anode current per length should be given by a formula like:

$$I/\ell \equiv j \propto \frac{k}{b} V_A^\frac{2}{3}$$  \quad (26)
although we could have an arbitrary function of the radius ratio: $b/a$ on the right-hand-side.

From Poisson’s equation we have:

$$\nabla^2 V = \frac{J}{\epsilon_0 v(r)} = \frac{I}{2\pi r \epsilon_0 v(r)} = \frac{j}{2\pi r \epsilon_0 \sqrt{\frac{2e}{m}}} V^{-\frac{1}{2}}$$

(27)

Using the Laplacian in cylindrical coordinates we find:

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} = \frac{j}{2\pi r \epsilon_0 \sqrt{\frac{2e}{m}}} V^{-\frac{1}{2}}$$

(28)

There is no known formula for the solution to this differential equation, but we can make considerable progress by writing the differential equation in terms of dimensionless quantities:

$$\frac{r}{a} = \rho$$

$$V = \left[ \frac{ja}{2\pi \epsilon_0 \sqrt{\frac{2e}{m}}} \right]^\frac{1}{2} f(\rho)$$

(30)

yielding:

$$\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} = f''(\rho) + \frac{1}{\rho} f'(\rho) = \frac{1}{\rho} f^{-\frac{1}{2}}$$

(31)

with initial conditions:

$$f(1) = 0$$

(32)

$$f'(1) = 0$$

(33)

We can numerically solve this differential equation using Mathematica:

```mathematica
NDSolve[{f''[\rho]+f'[\rho]/\rho==1/(\rho Sqrt[f[\rho]])}, f[1]==0, f'[1]==0, {f},{\rho,1,200}]
```

It’s actually not quite that simple. The cathode, at $\rho = 1$, is actually at a singular point of the differential equation (i.e., $f''(1) = \infty$). However the situation very near the cathode is well approximated by the planar case, where we’ve shown:

$$V(x) = \left[ \frac{9J_A}{4\epsilon_0 \sqrt{\frac{2e}{m}}} \right]^\frac{1}{2} x^\frac{5}{2} = \left[ \frac{9I}{2\pi a \epsilon_0 \sqrt{\frac{2e}{m}}} \right]^\frac{1}{2} (r-a)^\frac{5}{2} = \left[ \frac{9ja}{2\pi 4\epsilon_0 \sqrt{\frac{2e}{m}}} \right]^\frac{1}{2} \left( \frac{r-a}{a} \right)^\frac{5}{2}$$

$$= \left[ \frac{9}{4} \right]^\frac{1}{2} \left[ \frac{ja}{2\pi \epsilon_0 \sqrt{\frac{2e}{m}}} \right]^\frac{1}{2} \left( \frac{r-a}{a} \right)^\frac{5}{2}$$

(34)

So, near the cathode (i.e., $\rho$ slightly larger than 1):

$$f(\rho) \approx \left[ \frac{9}{4} \right]^\frac{1}{2} (\rho - 1)^\frac{5}{2}$$

(35)

We can use this approximation to start our numerical differential equation solution at a non-singular point (like $\rho = 1.00001$).
Figure 3: The plot on the left displays the dimensionless voltage $f$ obtained by numerical solution to the differential equation. The plot on the right compares various approximations for $f$ to the numerical solution.

Real devices are designed with $b/a \gg 1$. The behavior of $f$ for large $\rho$ can be determined by finding $A$ and $\alpha$ for which $f = A\rho^\alpha$ is a solution to the differential equation. One finds:

$$f = \left(\frac{9}{4}\rho\right)^{\frac{3}{2}}$$

A useful approximation for the range: $100 < b/a < 1000$ is:

$$f = \left(\frac{9}{4}\rho\right)^{\frac{3}{2}} + 2$$

(For example, the device used in lab has $b/a = 121.5$. For this value, the differential equation gives $f = 44.136$; the above approximation gives: $f = 44.130$.)

We recover Child’s law by rearranging (30):

$$\frac{2\pi\varepsilon_0 \sqrt{\frac{2e}{m}}}{a} \left[ \frac{V_A}{f(b/a)} \right]^{\frac{3}{2}} = j = I/\ell$$

(Note: Langmuir’s original work (Phys. Rev. 22, 347 (1923)) on this subject is expressed in terms of $\beta$ where:

$$\beta^2(\rho) \equiv \frac{4}{9} \frac{f^\frac{3}{2}}{\rho} = \begin{cases} \rho^{-1} & \rho \to 1 \\ \rho^{-\infty} & \rho \to \infty \end{cases}$$

So:

$$\frac{8\pi\varepsilon_0 \sqrt{\frac{2e}{m}}}{9b\beta^2} V_A^\frac{3}{2} = j = I/\ell$$

$\beta^2 = 1.072$ for the device used in lab.

Richardson’s Law

Most any thermal process is governed by the Boltzmann factor:

$$e^{-\frac{\Delta E}{kT}}$$
where $k$ is the Boltzmann constant. Approximately speaking the Boltzmann factor expresses the relative probability for an event requiring energy $\Delta E$ in a system at (absolute) temperature $T$. Clearly if $\Delta E \gg kT$, the probability of the event happening is low. If an electron requires an energy $W$ (called the work function) to escape from the metal, The Boltzmann factor suggests that this would happen with relative probability $e^{-W/kT}$. Thus you should expect that the current emitted by a heated metal would follow:

$$I \sim e^{-\frac{W}{kT}} \quad (42)$$

Clearly you should expect different elements to have different work functions, just as different atoms have different ionization potentials. What is surprising is that the proportionality factor in the above equation includes a universal constant — that is, a constant that just depends on the properties of electrons (and, most importantly, Planck’s constant, $\hbar$) and does not depend on the type of material. That is to say that thermionic emission probes the quantum state of the material statistically, whereas the photoelectric effect probes much the same physics electron by electron. (The next level problem is to explain why this universal constant (the Richardson constant, $A$) in fact does depend a bit on the material.) To show:

$$J = AT^2 e^{-W/kT} \quad (43)$$

where

$$A = \frac{4\pi e m k^2}{\hbar^3} = 1.2 \times 10^6 \text{A/m}^2\text{K}^2 \quad (44)$$

Quantum Theory: Free Electron Gas

Instead of thinking about electron particles bouncing around inside a box, de Broglie invites us to consider standing waves of electron probability amplitude:

$$\psi = N \exp(ik_x x) \exp(ik_y y) \exp(ik_z z) = Ne^{ik \cdot r} \quad (45)$$

Recall that vector $\hbar k$ is the momentum, $p = mv$, of the electron and $\hbar = \hbar/2\pi$. Periodic boundary conditions on the box (which we take to be a cube with one corner at the origin and the diagonally opposite corner at the point $r = (L, L, L)$) require each component $k_i$ to satisfy:

$$k_i = \frac{2\pi n_i}{L} \quad (46)$$

where each $n_i$ is an integer. Thus each wave function is specified by a triplet of integers: $n = (n_x, n_y, n_z)$, the $\mathbf{n}$-vector. Applying Schrödinger’s equation, we find that this wavefunction has energy:

$$E(n) = \frac{\hbar^2 k^2}{2m} = \frac{(2\pi \hbar)^2 n^2}{2mL^2} = \frac{(2\pi \hbar)^2 (n_x^2 + n_y^2 + n_z^2)}{2mL^2} \quad (47)$$

Notice that there is a definite relationship between the velocity vector of the electron and the $\mathbf{n}$-vector.

$$\mathbf{v} = \frac{2\pi \hbar}{mL} \mathbf{n} \quad (48)$$

Another way of saying the same thing is that allowed quantum-state velocities form a cubic lattice with cube-side $2\pi \hbar/mL$. The number of states with electron velocities in some specified region (for example a parallelepiped with sides: $\Delta v_x \Delta v_y \Delta v_z$) can be found from...
Figure 4: Electrons in the metal experience a constant confining potential of depth $U$. Possible quantum mechanical states for these electrons are displayed as horizontal lines. Electrons fill all the available states up to the Fermi energy, $E_F$. The work function, $W$, is defined at the minimum energy needed to remove an electron from the metal. As shown above: $W = U - E_F$.

the number of $2\pi\hbar/mL$ sided cubes that fit into the region, which is the volume of that velocity-space region divided by $(2\pi\hbar/mL)^3$. Hence:

\[
\text{number of states with velocity between } \mathbf{v} \text{ and } \mathbf{v} + \Delta\mathbf{v} = \frac{\Delta v_x \Delta v_y \Delta v_z}{(2\pi\hbar/mL)^3} \quad (49)
\]

\[
\text{number of states per volume with velocity between } \mathbf{v} \text{ and } \mathbf{v} + \Delta\mathbf{v} = \frac{\Delta v_x \Delta v_y \Delta v_z}{(2\pi\hbar/m)^3}
\]

\[
= \left(\frac{m}{2\pi\hbar}\right)^3 \Delta v_x \Delta v_y \Delta v_z = \mathcal{N} \Delta v_x \Delta v_y \Delta v_z \quad (50)
\]

where $\mathcal{N}$ is the (constant) density of states in velocity space.

**Quantum Theory: Fermi Energy**

Fermions (half-integer spin particles), in contrast to bosons (integer spin particles), cannot group together. Since the electron is “spin $\frac{1}{2}$”, each of the above states can hold at most 2 electrons: one spin up and one spin down. The probability that a particular state with energy $E$ will be occupied is given by Fermi-Dirac statistics:

\[
f(E) = \frac{1}{1 + \exp\left(\frac{E - E_F}{kT}\right)} \quad (51)
\]

where $E_F$ is called the Fermi energy. The Fermi energy is basically a disguise for the number of electrons, as, approximately speaking, it is the dividing line between occupied states and unoccupied states. (If the Fermi energy is high, there must be lots of occupied states and hence lots of electrons.) Note that if $E \gg E_F$, the exponential factor is huge and we can neglect the “+1” so

\[
f(E) \approx \exp\left(-\frac{E - E_F}{kT}\right) \quad (52)
\]
Classical Theory: Electron Escape

The density of states combined with the Boltzmann factor gives us the number of free electrons per unit volume with a particular velocity. In order for an electron to escape during some time $\Delta t$, it must have $v_x$ sufficient to overcome the image-charge barrier and it must be sufficiently close to the wall. All the electrons with $v_x > \sqrt{2U/m}$ within a distance $v_x \Delta t$, will escape, where $U$ is the depth of the potential well for the electrons. Thus to calculate the number of electrons escaping through area $A$ during $\Delta t$ we must

\[
\int_{\sqrt{2U/m}}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z 2Nf(E)A v_x \Delta t
\]

\[
= 2N e^{E_F/kT} A \Delta t \int_{\sqrt{2U/m}}^{\infty} e^{-mv_y^2/2kT} v_y dv_y \int_{-\infty}^{\infty} e^{-mv_x^2/2kT} v_x dv_x \int_{-\infty}^{\infty} e^{-mv_z^2/2kT} dv_z
\]

\[
= \frac{4\pi m(kT)^2}{(2\pi \hbar)^3} A \Delta t \exp \left( -\frac{(U - E_F)}{kT} \right)
\]

The electric current density is the electric charge escaping per time per area:

\[
J = \frac{\text{number escaping}}{A \Delta t} = \frac{4\pi e m(kT)^2}{h^3} \exp \left( -\frac{W}{kT} \right)
\]

which is Richardson’s equation, with work function $W$ given by $U - E_F$.

Experiment: Richardson’s “Constant” is Material Dependent

Experimentally it is found that Richardson’s constant depends on the material. Why?
Figure 6: When an electric field $E$ is present, the total potential outside of the metal will be the sum of the image-charge potential and the electric field potential ($-\varepsilon E x$). The result is a reduction in the energy needed to escape the metal. If the classically disallowed barrier is sufficiently thin, quantum mechanical tunneling through the barrier can be measured.

1. The work function depends on temperature (due to, for example, thermal expansion). If the data analysis assumes it’s constant, the resulting $A$ will be grossly in error.

2. Classically reflection requires a turning point (where $v_x = 0$), whereas quantum mechanical reflections are possible just due to sharp changes in potential. Quantum mechanical reflection at the metal boundary was not included in our calculations; we assumed every energetic electron headed toward the wall went through the wall.

3. Surface contamination can affect emission probability. In fact, it was originally thought that thermionic emission was 100% due to surface contamination. (You can think of surface contamination as a locally varying work function.)

4. Even in the absence of surface contamination, in typical experiments, the metal is polycrystalline and different crystal surfaces have different work functions.

5. Real experiments may only approximate the idealized theory. For example, since the hot cathode needs to be supported and heated, it is generally not at a uniform temperature.

**Schottky Emission**

For “small” anode voltages, the space charge partially shields the cathode, and the current is specified by Child’s law. At higher voltages, the anode collects every electron emitted, and the current is specified by Richardson’s law. At higher fields yet, the strong electric field reduces the barrier to emission effectively reducing the work function. This is Schottky emission. At very strong fields the electrons can quantum mechanically tunnel through the barrier even though they lack the energy required classically to overcome the image-charge
force. We seek to estimate the extent to which Schottky emission affects our determination of the work function.

In the presence of an electric field, the potential near the cathode is given by:

$$U(x) = -\frac{e^2}{4\pi\varepsilon_0 4x} - eEx$$  \hspace{1cm} (55)

where $E$ is the electric field. We seek the amount the electric field reduces the barrier-top potential. The maximum potential is found by finding where $dU/dx$ is zero:

$$\frac{dU(x)}{dx} = \frac{e^2}{4\pi\varepsilon_0 4x^2} - eE = 0 \iff x_m = \sqrt{e/16\pi\varepsilon_0 E}$$  \hspace{1cm} (56)

The potential at this maximum is:

$$U(x_m) = -\sqrt{\frac{e^3E}{4\pi\varepsilon_0}}$$  \hspace{1cm} (57)

For most vacuum tubes this shift in work function is “small”. However, microscopic peaks on the surface of the metal will enhance the electric field near the peak beyond that calculated for a smooth surface resulting in locally enhanced Schottky emission.

**Experiment**

This experiment involves thermionic emission from the hot tungsten filament of a G.E. FP-400 vacuum tube.

**Temperature Determination**

Often temperature measurement in physics experiments is easy. In the “normal” range of temperatures there are many types of temperature transducers which convert temperature to an electrical quantity (e.g., Pt resistance thermometers, thermocouples, thermistors, ICs). However at the extremes of high and low temperature, measurement becomes tricky. Questions like “What exactly defines temperature?” must then be answered. This experiment requires “high” temperatures in a vacuum, so we do not have direct contact with the
material whose temperature we seek. In addition the FP-400 tube was not built by us, so we have limited direct information about the device.

One common way to measure temperature is by using the temperature dependence of resistance. The resistance of metals is approximately proportional to temperature. Jones and Langmuir\(^1\) have published a table of resistance vs. temperature for tungsten filaments, from which Kirkman has found an approximating formula:

\[
T_r = 112 + 202x - 1.81x^2
\]  

where \(x\) is the ratio of the hot resistance to that at 293 K.

The problem with this approach is that the measured resistance, \(R_{\text{measured}}\), will include both the resistance of the filament and the wires supporting it in the vacuum tube. (Take care that your measurement of it does not include further complicating factors, like the resistance in the external wires.) Unfortunately the wire-support resistance, \(R_{\text{support}}\), is larger than the filament resistance at low temperatures. Thus the quantity we seek (filament resistance, \(R_W\)) must be calculated as the small difference between two numbers:

\[
R_W = R_{\text{measured}} - R_{\text{support}}
\]  

a situation that produces big relative errors. Additionally, we have no independent way of measuring \(R_{\text{support}}\) (we can’t take the tube apart), so we can only calculate it based on a calculated value for \(R_W\) at room temperature. (This calculated value for \(R_W\) will depend on the spec sheet values for the dimensions of the filament and the resistivity of W-filament material at room temperature: \(\rho_{293} = 5.49 \, \mu\Omega \cdot \text{cm}\).) You should be quite uncomfortable that so much of this measurement depends on “book values” rather than things you can directly check. In addition we have no choice but to assume that the found \(R_{\text{support}}\) is a constant. This is surely false as the filament will heat the ends of the support, and the resistance of essentially every material depends on temperature.

There is a further problem with any measurement of voltage when parts of the system are at different temperatures: thermally induced emfs (thermocouples). If the ends of the tungsten filament are at different temperatures, there will be a voltage induced approximately proportional to the temperature difference between the two ends. This additional voltage

\(^1\)GE Rev 30, 310 (1927)
source confuses the resistance determination. The phenomena can be detected and corrected by reversing the direction of current flow (which reverses the Ohm's law voltage, but does not affect the sign of the thermal voltage.) Thermal voltages are generally less than a mV, and so are negligible once our measured voltages approach a Volt.

Another approach is suggested by Jones & Langmuir. In a vacuum the temperature of a current-carrying wire is the result of an equilibrium between electrical power dissipated in the wire and energy lost in the form of radiation. (We assume that energy lost through conduction — either through the wire-supports or residual air in the “vacuum” — is negligible.) According to the Stefan-Boltzmann law, the power radiated from the surface of a hot black-body is given by:

$$P = \sigma T^4 A$$

where $\sigma$ is the Stefan-Boltzmann constant, $T$ is the temperature of the body, and $A$ is the surface area of the body. (In fact tungsten is not a black-body, so when applied to tungsten the above should be multiplied by a “fudge factor”, the total emissivity $\epsilon_T$, about 0.3.) Using just crude approximations, we equate the electrical power dissipated to the power radiated:

$$I^2 T \frac{\ell}{d^2} \sim I^2 T \frac{\ell}{4d^2} \sim I^2 R = \epsilon_T \sigma T^4 \pi d \ell \sim T^4 d \ell$$

where $d$ is the diameter of the wire and $\ell$ is the length of the wire. On the right hand side we’ve assumed that the resistivity of tungsten is proportional to temperature, and on the left hand side we’ve assumed $\epsilon_T$ doesn’t depend on temperature. We conclude:

$$I^2 \sim T^3 d^3$$

Figure 8: The temperature dependence of tungsten-filament resistance from the data of Jones & Langmuir with an approximating curve. The x-axis is the ratio of the hot resistance to that at 293 K.
Figure 9: The temperature of an in-vacuum tungsten filament as a function of current from the data of Jones & Langmuir with an approximating curve. The x-axis is the current divided by the diameter of the wire (in cm) raised to the $\frac{3}{2}$ power.

\[
\left( \frac{I}{d^{\frac{3}{2}}} \right)^{\frac{2}{3}} \sim T \quad (63)
\]

Absent the above approximations we can hope that temperature is a function of $a \equiv I/d^{\frac{3}{2}}$. Once again Jones & Langmuir provide us with calibrating data for this expected relationship. For temperatures $400 \, \text{K} < T < 3000 \, \text{K}$, Kirkman finds:

\[
T_i = 117 + 56a^{0.5} + 0.00036a^{1.8} \quad (64)
\]

Finally, attaining thermal equilibrium is a problem that affects most any temperature measurement. The balance between electrical power in and heat lost is not immediately achieved. The parts of the system with large heat capacities (particularly the filament supports and other large structures in the tube), will only gradually approach equilibrium. Thus “soak” time is required following each jump in heating power. The effect of this “thermal inertia” is seen in “hysteresis”: temperatures obtained while increasing the temperature disagree with those found while decreasing the temperature. This will be an important source of uncertainty.

**Electrical Measurements**

As shown in Figure 7, the filament is powered by a Keithley 2420 source-meter. The filament voltage is measured directly at the socket with a Keithley 192 voltmeter. By combining a current measured from the 2420 with the voltage from the 192, the resistance...
of the tube alone an be determined. Of course, only part of the tube resistance is filament resistance. The room temperature filament resistance \( R_W \) can be calculated from the specifications of the FP-400 tube. Using a four-terminal measurement, measure the room temperature resistance of the tube \( R_{\text{measured}} \). Confirm this measurement by sourcing 1 mA and 10 mA into the tube and reading the resulting voltage on the 192. Use these measurements to determine a value for \( R_{\text{support}} \). At the same time the resistance of the external wires can be found. The voltage difference between the 192 and the 2420 must be dropped across the wires. From the voltage across the external wires and the current flowing through them, calculate the wire resistance. Use this resistance to calculate the 2420 voltage limit. (That is the voltage the 2420 would need to produce to have the maximums (4.75 V, 2.5 A) on the filament.)

You will be collecting two types of data at the same time: thermal characteristics of the filament and the thermionic properties of the tube. Starting at a filament current of 0.9 A, increase the current flowing through the filament in steps of 0.1 A to a near-maximum current (e.g., 2.4 A) and then reverse those steps down to 1.0 A. (Note the maximum limits of the filament: 2.5 A or 4.75 V. Do not exceed either! You will probably need to do some experimenting to determine the maximum current you can use without exceeding the 4.75 V limit.) The up-sweep in filament current followed by the down-sweep will allow you to test for hysteresis. At each step in current allow the filament to approach thermal equilibrium (wait, say, 15 seconds) and then measure the voltage across and current through the tube. Calculate filament temperature two ways (Equations (58) and (64)). Average the two to estimate the temperature, and use half the absolute value of the difference to estimate the error.

By conservation of energy we know that the power dumped in the wire (mostly from electrical heating, but also from other sources like radiation from the room temperature environment to the wire) should equal the power out of the wire (from black-body radiation and other factors like conduction down the wire supports). Thus:

\[
\epsilon_{T,\sigma}AT^4 = I^2R_W + \text{constant} \quad (65)
\]

\[
T^4 = \frac{1}{\epsilon_{T,\sigma}} I^2R_W + \text{constant} \quad (66)
\]

\[
y = ax + b \quad (67)
\]

A graph of \( T^4 \) vs. power should be a straight line from which you will determine \( \epsilon_{T,\sigma} \). (Note that the error in power is quite small, so we have properly used it as an \( x \) variable.) In order to test this relationship you will want to make a file containing the filament power, \( T^4 \) (use the average of the two temperatures: \( (T_i^4 + T_r^4)/2 \)), and the error in \( T^4 \) (use half the difference from the two temperatures: \( |T_i^4 - T_r^4|/2 \)).

In the second part of the experiment, you will collect anode (voltage, current) curves for each different cathode temperature. Use the Keithley 2400 to sweep the anode voltage logarithmically from 1 V to 120 V. (Note the maximum limits for the anode: 0.055 A or 125 V. Do not exceed either!) According to Child’s law, the anode current, \( I_A \), should increase in proportion to \( V_A^2 \). At sufficiently high voltage the current will be limited by the maximum electron evaporation rate given by Richardson’s law. At the maximum temperature (corresponding to the maximum filament current) you will want to make a file containing \( V_A \), \( I_A \), and \( \delta I_A \) which you can use to fit to the Child’s law functional form:

\[
I_A = k_1(V_A - k_2)^{3/2} \quad (68)
\]
In addition, you will want to make a big continuous file containing: $V_A$, $I_A$ at every temperature tested.

**Computer Data Collection**

As part of this experiment you will write a computer program to control the experiment. *Plagiarism Warning:* like all lab work, this program is to be your own work! Programs strikingly similar to previous programs will alarm the grader. I understand that this will often be a difficult and new experience. Please consult with me as you write the program, and test the program (with tube disconnected!) before attempting a final data-collecting run.

Your program will control all aspects of data collection. In particular it will:

1. Declare and define all variables.

2. Initialize the source-meters which must be told maximum voltage and current to occur during the experiment. In particular you will need to know the external wire resistance to estimate the maximum voltage that the 2420 can safely output at maximum current and keep the tube voltage less than the maximum rating.

3. Display the status of all devices before starting data collection.

4. Open files:
   - (a) `filament.dat` for $I, V, T_r, T_i$ of filament.
   - (b) `stefanB.dat` for power, $T^4, \delta T^4$ of filament.
   - (c) `VI.dat` for $V_A, I_A$ of anode, with notes for filament $I, T_r, T_i$
   - (d) `child.dat` for $V_A, I_A, \delta I_A$ of anode at maximum filament current.

5. Start an up-sweep in the filament current: 0.9 A to maximum (e.g., 2.4 A).
   - (a) Request the 2420 source-meter to source a filament current.
   - (b) Let the system sleep for 15 seconds to approach thermal equilibrium.
   - (c) Request a logarithmic anode voltage sweep from 1 V to 120 V.
   - (d) Turn off the anode voltage.
   - (e) Read the source-meter to get the exact filament current (it will be very close to the requested current).
   - (f) Read the 192 to get the exact voltage across the tube.
   - (g) Calculate the two filament temperatures. (Requires knowledge of $R_W$ and $R_{support}$ and use of Kirkman formulas.)
   - (h) Add a line to the file: `stefanB.dat` containing filament power, $T^4$, and $\delta T^4$.
   - (i) Add a line to the file: `filament.dat` containing $I, V, T_r, T_i$ of the tube.
   - (j) Add a comment (!) line to the file: `VI.dat` containing $I, T_r, T_i$ of filament.
   - (k) Record the anode sweep data in the file: `VI.dat`.
   - (l) Increment the filament current and return to (a).
Figure 10: The temperature dependence of thermionic emission in a FP-400 vacuum tube. Each curve shows the anode current-voltage relationship at a particular filament temperature. At high filament temperatures and low anode voltages the anode current follows Child’s law, an upward sloping straight line on this log-log plot. At sufficiently high anode voltage, the filament current plateaus at a value given by Richardson’s law. At low filament temperatures and high anode voltages, “cold emission” is the dominant effect.

6. Take an additional special sweep at the maximum filament current, following all the steps outlined above. In addition record the anode sweep data in the file `child.dat`.

7. Start a down-sweep in the filament current: 2.3 A to 1.0 A, following all the steps used in the up-sweep.

8. Turn off the output of the 2420.

9. Close all files.

Note that the 0.9 A filament current data is just trash collected so the 1.0 A filament current data is taken on a pre-warmed filament.

Data Analysis

The main result of this experiment is a plot similar to Figure 10 showing the anode current-voltage relationship at various filament temperatures. Note that each curve is a pair: one from the up-sweep in filament temperature, one from the down-sweep. See that hysteresis is largest at low filament temperature. Note that for the lowest filament temperatures, no current plateau is in evidence. Starting at about $I_f = 1.2$ A, plateau currents can be
Figure 11: A Richardson Plot of thermionic emission in a FP-400 vacuum tube. Each data point displays the plateau anode current at a particular filament temperature. The curve through the data fits for the work function; the slightly steeper curve uses the book value for the work function.

determined. According to Richardson’s law, these plateau currents should satisfy:

\[ I = AAT^2 e^{-B/T} \]  

(69)

where \( A \) is the tungsten filament area. The errors in \( I \) can be estimated from the hysteresis, but in fact the largest errors are in temperature. A fit to the above relationship allows us to estimate the work function, expressed in units of temperature. Convert this value to joules and eV and compare to the book value of 4.5 eV. Since our temperatures are so uncertain, particularly at the low end, the best estimate for the Richardson constant \( A \) comes from substituting the known value of the work function and the data for the highest filament temperature into the above equation. Do this and compare to the book value: \( A = 0.72 \times 10^6 \text{ A/m}^2\text{K}^2 \). See Figure 11 for typical results.

Examine the relative importance of hysteresis and calibration in temperature determination. For \( I_f = 1.2 \text{ A} \) record the difference in \( T_r \) due to hysteresis and the difference between \( T_r \) and \( T_i \) which is a temperature calibration uncertainty.

At sufficiently low anode voltage (and high filament temperature), Child’s law governs the anode current-voltage relationship: Equation (68). Fit the highest anode-temperature data (e.g., filament current of 2.4 A) to this functional form. From the constant \( k_1 \) the electron charge-mass ration \( e/m \) can be found. Since the FP-400 is a finite length cylinder (rather than the infinite cylinder discussed in our theory) use the effective length = \( 0.7 \times \ell \) as the length of the cylinder. Compare to book values.

\[^2\]Blakemore, Solid State Physics
Figure 12: A plot of the space-charge limited thermionic emission in a FP-400 vacuum tube (Child’s law). The data was taken at a filament current of 2.4 A. Every other data point has been eliminated so the fit line is not obscured. Note that the fit line systematically misses the data, sometimes a bit high others a bit low. Nevertheless, the law provides an excellent summary of the data over a huge range of variation.
Figure 13: A test of the Stefan-Boltzmann law: that energy radiated is proportional to $T^4$. Note that the fit line hits well within each error bar. The $\chi^2$ for this fit will be “small”. Evidently the average temperature is a better measure of temperature than you might expect from the deviations between $T_i$ and $T_r$.

The Stefan-Boltzmann relationship provides a check on our temperature calibration (in fact it is the definition of temperature beyond the melting point of Pt). Fit the data in stefanB.dat to Equation (67). Determine $\epsilon_T$ from the fit slope.

**Report Checklist**

1. Calculations (no errors) of $R_W$. Measurements (4-wire ohmmeter and direct voltage/current) of $R_{\text{measured}}$. Calculation of $R_{\text{support}}$. Calculation of external wire resistance.

2. Data files and computer program. Leave them in your UNIX account. Print out a copy of your program and tape it into your lab notebook.

3. Plots similar to Figures 10, 11, 12, 13. Note that Figure 10 is complex to produce. Use a file of plot commands and feel free to talk to me about how to go about doing this. Carefully note the use of log and inverse scales. Also include a fit report for each fit curve.

4. Experimental values for: $W$ (in eV), $A$, $e/m$ and $\epsilon_T$. Calculate error only in $\epsilon_T$.

5. Show the missing steps leading to Equations (31) and (36). Substitute the $\rho \rightarrow 1$ approximation Equation (35) into the differential equation Equation (31). Show that while we do not have an exact solution to the differential equation, the singular parts (i.e., those that approach infinity as $\rho \rightarrow 1$) cancel.
6. Calculate the electric field at the surface of the FP-400 cathode for $V_A = 120$ V. (Assume zero space charge for this calculation, i.e., that the vacuum has been swept clean of electrons.) Calculate the shift in work function (in eV) due to Schottky effects for this field.

7. Show how the potential energy Equation-(55) is consistent with the image charge force Equation-(1).

Comment: Classical vs. Quantum

I said above that the presence of an $h$ in Richardson’s constant shows that the process is governed by quantum mechanics. It is not quite so simple. The evaporation of water is not fundamentally different from that of electrons, while the former (due to the larger mass of a water molecule) is a classical process. Classical evaporation can be calculated much as quantum evaporation using the Maxwell-Boltzmann speed distribution and the number density (of water molecules) rather than the disguised version of this: Fermi energy ($E_F^{3/2} \propto$ number density). We can write the classical rate in terms of the quantum with no $h$ visible:

$$\text{classical flux} = \frac{4}{3\sqrt{\pi}} \left[ \frac{E_F}{kT} \right]^{3/2} \times \text{quantum flux} \quad (70)$$

The different temperature dependence$^3$ for the classical flux ($AT^{4}e^{-W/kT}$ vs. $AT^{2}e^{-W/kT}$) cannot be detected experimentally: the Boltzmann factor overwhelms all other temperature dependences. On the other hand, since $E_F \gg kT$, the classical rate comes out much larger than the quantum rate. This played a role in the mistaken idea that thermionic emission was due to surface contamination: the experimental rate seemed too small to be thermal evaporation. On the other hand a more fruitful interpretation of the “low” rate was that only a fraction ($\sim kT/E_F$) of the electrons were thermally active. This connected with other observations (like the “small” specific heat of the electron gas) and provided a link to the idea of a degenerate Fermi gas.

References

5. Koller *The Physics of Electron Tubes*, 1937, Ch. I, VI, VIII

$^3$Saul Dushman (*Phys. Rev.* 21, 623–636 (1923)), while working at G.E., provided a general thermodynamic argument for the $T^2$ dependence and the universal nature of $A$. The resulting relationship is therefore sometimes known as the Richardson-Dushman relationship.